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In 2004/2005, the course WISB342 was given by Prof. dr. D. Siersma.

Differentiable Manifolds (WISB342)
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Excercise 1

Consider the 2-dimensional real projective plane $\mathcal{P}^2(\mathbb{R})$. Points can be described by ratio's $[x : y : z]$. One can take 3 coordinate patches $\bigcup_x = \{[x : y : z]|x \neq 0\}$; $\bigcup_y$ and $\bigcup_z$ similar.

a. Describe charts $\bigcup_x \to \mathbb{R}$ and $\bigcup_y \to \mathbb{R}^2$ and compute the transition function.

b. Let S^2 be the 2-sphere in $\mathbb{R}^3$ given by $x^2 + y^2 + z^2 = 1$ and $f : S^2 \to \mathcal{P}^2(\mathbb{R})$ be griven by $(x, y, z) \to [x : y : z]$. Choose a coordinate patch for S^2 and one for $\mathcal{P}^2$ and describe f on the choosen charts.

Excercise 2

Let be given the smooth* manifolds M, N and P and the smooth maps $f : M \to N$ and $G : N \to P$

a. Show that $g \circ f : M \to P$ is a smooth map (starting from the definition on charts).

b. Give the definition of tangent vector $X \in T_pM$ (in terms of the equivalence classes of curves) and show that $D_p(g \circ f) : T_pM \to T_{gfp}P$ is equal to the composition $D_{fp}(g) \circ D_p(f)$

Excercise 3

Let V and W be vectorfields on a manifold M and let f, f_1, f_2, g be functions on M. Show:

a. $[f_1V, f_2W](g) = f_1f_2[V, W](g) + f_1V(f_2)W(g) - f_2W(f_1)V(g)$

b. $[V, W](f \cdot g) = g \cdot [V, W](g) + f \cdot [V, W](g)$

Excercise 4

Let $M = \mathbb{R}^2$. We consider for $t \in \mathbb{R}$ and $s \in \mathbb{R}$ the following 1-parameter families of maps:

$$\begin{cases} H_t(x, y) &= (x + t, y) \\ K_s(x, y) &= (x, y + sx) \end{cases}$$

a. Show that $\{H_t\}$ and $\{K_s\}$ satisfy the definition of flow.

b. Compute the infinitesimal generators V of $\{H_t\}$, resp. W of $\{K_s\}$

c. Compute $K_{-s}H_{-t}K_sH_t(x, y)$

d. Let f be any function on $\mathbb{R}^2$. Compute $[V, W](f)$ and give an expression for $[V, W]$ in terms of $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$

⌜N.B. If you are not sure about your answers in b) then you may use $V = 2\dfrac{\partial}{\partial x}$ and $W = 3\dfrac{\partial}{\partial x}$⌟

e. Compute the infinitesimal generator of $K_{-t}H_{-t}K_tH_t$

Excercise 5

Let $s : V \to V$ be a linear map between 3-dimensional vectorspaces, given by:

$$\begin{cases} s(e_1) & = & e_1 \\ s(e_2) & = & 2e_1 + 4e_2 \\ s(e_3) & = & 3e_1 + 5e_2 + 6e_3 \end{cases}$$

a. Compute the matrix of $\bigwedge_s^2 : \bigwedge^2 V \to \bigwedge^2 V$ (wrt $e_i \wedge e_j | i < j$)

b. Compute a matrix of $\bigwedge_s^3 : \bigwedge^2 V \to \bigwedge^3 V$

c. Identify $\bigwedge_s^4 : \bigwedge^4 V \to \bigwedge^4 V$