Department of Mathematics, Faculty of Science, UU.
Made available in electronic form by the $\mathcal{T_BC}$ of A–Eskwadraat
In 2005/2006, the course WISB342 was given by D. Roytenberg.

Differentiable Manifolds (WISB342)
1 February 2006

- For the full examination: Exercises 1, 2, 3, 4, 5.
- For the second part of the examination: Exercises 3, 4, 5, 6, 7.

Question 1

Recall that points on the real projective plane $\mathbb{R}P^2$ can be described by ratios $[X : Y : Z]$. Consider the map $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ given by $f([X : Y : Z]) = [X^2 : Y^2 : Z^2]$.

a) Show that f is well-defined and describe it in each of the three inhomogeneous coordinate charts ($[X : Y : Z] \to (Y/X, Z/X)$ for $X \neq 0$, and similarly for the other two).

b) Describe the subset of $\mathbb{R}P^2$ consisting of the critical points of f.

Question 2

Consider the following vector fields on $\mathbb{R}^2$:

$$V_1 = Y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}; \quad V_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}; \quad V_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

a) Work out the commutators $[V_i, V_j]$.

b) Describe the flow of each V_i.

c) Show that every sphere centered at the origin is left invariant by all three flows. Is there a smaller subset of such a sphere with this property?

Question 3

We consider the torus T^2 embedded intro $\mathbb{R}^3$ as follows:

$$x = (2 + \cos\theta^1)\cos\theta^2; \quad y = \sin\theta^1; \quad z = (2 + \cos\theta^1)\sin\theta^2$$

for $\theta^1 \in [-\pi/2, 3\pi/2)$, $\theta^2 \in [-\pi, \pi)$. Consider the map $G : T^2 \to S^2$ which assigns to every $p \in T^2$ the outward pointing normal vector at p.

a) Describe the set $G^{-1}(\vec{n})$, where $\vec{n} = (0, 0, 1)$ is the north pole, and show that $\vec{n}$ is a regular value of G.

b) Compute the degree of G.

Question 4

The Künneth formula for the de Rham cohomology reads:

$$H^k(M \times N) = \bigoplus_{i+j=k} H^i(M) \otimes H^j(N).$$

a) Use the formula to compute the dimension of $H^k(T^3)$ for $k = 0, 1, 2, 3$, where $T^3 = S^1 \times S^1 \times S^1$ is the 3-torus.

b) Find explicit closed but not exact 1-forms ϕ_i corresponding to a basis of $H^1(T^3)$.

c) Find closed but not exact 2-forms ψ^j such that $\int_{T^3} \phi_i \wedge \psi^j = \delta_i^j$, corresponding to the Poincaré dual basis for $H^2(T^3)$.

Question 5

Let $c : [0,1]^2 \to \mathbb{R}^3 - \{0\}$ be given by $c(s,t) = (\sin \pi t \cos 2\pi s, \sin \pi t \sin 2\pi s, -\cos \pi t)$, and let $\omega = (xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)/r^3$, where $r = \sqrt{x^2+y^2+z^2}$.

a) Show that $\partial c = 0$ and $d\omega = 0$.

b) Compute $\int_c \omega$. Is ω exact? Is c a boundary? Justify your answers.

c) Why is $\int_c \omega = \int_{S^2} \iota^* \omega$, where ι is the inclusion of the unit sphere?

End of the full exam

Question 6

Let M be an n-dimensional manifold, X a vector field on M, ω a nowhere vanishing n-form.

a) Show there exists a smooth function $f_{X,\omega}$ such that $L_X \omega = f_{X,\omega} \omega$.

b) Suppose $f_{X,\omega}$ vanishes identically. What is the relation between ω and $\phi_t^* \omega$, where ϕ_t is the flow of X? If $U \subset M$ is a compact n-dimensional manifold with boundary, what is the relation between $\int_U \omega$ and $\int_{\phi_t(U)} \omega$?

c) Let $M = \mathbb{R}^3$, $X = (A, B, C)$, $\omega = dx \wedge dy \wedge dz$. Compute $f_{X,\omega}$ and relate it to a well-known quantity in vector calculus.

Question 7

What is the de Rham cohomology of the Möbius band? Justify your answer.