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In 2005/2006, the course WISB342 was given by Dmitry Roytenberg.

Differentieerbare Variëteiten (WISB342)
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Question 1

Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$ given by $F(x, y, z) = x^2 + y^2 - z^2$.

a) For which values r is $M_r = F^{-1}(r)$ a manifold? Why? What is its dimension? How many connected components does M_r have, depending on r? Sketch a picture of M_r for several typical values of r.

b) Find an atlas for M_1 consisting of two charts and compute the transition map between them (*Hint*: use cylindrical coordinates).

c) Show that M_1 is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.

Question 2

Consider the following vector fields on $\mathbb{R}^3$:

$$V_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}; \quad V_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}; \quad V_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

a) Show that all the V_i's are tangent to the unit sphere S^2, and that their values span T_pS^2 at every $p \in S^2$.

b) Show that, nevertheless, no two of the three V_i's suffice to give a basis for T_pS^2 at every p.

c) Find smooth functions c^1, c^2 and c^3 on $\mathbb{R}^3$ such that $c^iV_i = 0$ identically on $\mathbb{R}^3$.

Question 3

Let M be a manifold, V and W vector fields on M. Consider the operator $[V, W] : C^\infty(M) \to C^\infty(M)$ defined by $[V, W](h) = V(W(h)) - W(V(h))$.

a) Show that for $f, g \in C^\infty(M)$,

$$[fV, gW] = fg[V, W] + fV(g)W - gW(f)V$$

b) Show that $[V, W]$ is in fact a derivation, hence a vector field whose value at $p \in M$ is given by $[V, W]_p(h) = V_p(W(h)) - W_p(V(h))$.

c) If $V = v^i\frac{\partial}{\partial x^i}, W = w^j\frac{\partial}{\partial x^j}$ in some coordinate chart (x, U), with $v^i, w^j \in C^\infty(U)$, it follows that $[V, W] = c^k\frac{\partial}{\partial x^k}$ for some $c^k \in C^\infty(U)$.

Express the c^k's in terms of the v^i's and w^j's. In particular, what is $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$?

Question 4

Let M be a manifold, $h \in C^\infty(M)$.

a) Show that $p \in M$ is a critical point of h if and only if $v(h) = 0$ for all $v \in T_pM$.

b) For a critical point p of h and $v, w \in T_pM$, define $H_{h,p}(v, w) = v(\tilde{w}(h))$, where $\tilde{w}$ is a vector field defined in some neighborhood of p whose value at p is w. Show that $v(\tilde{w}(h)) = w(\tilde{v}(h))$ (where $\tilde{v}$ is, likewise, an extension of v to a vector field near p), and deduce from this that the definition of $H_{h,p}$ only depends on v and w rather than their extensions. Thus, $H_{h,p} : T_pM \times T_pM \to \mathbb{R}$ is a well-defined symmetric bilinear form, known as the *Hessian* of h at p. A critical point is called *nondegenerate* if the matrix $H_{ij} = H_{h,p}(e_i, e_j)$ of the Hessian with respect to some (hence any) basis $\{e_i\}$ is nonsingular. The *index* of a nondegenerate critical point is, by definition, the number of negative eigenvalues of the Hessian at that point.

c) Consider the torus T^2 embedded in $\mathbb{R}^3$ as follows:

$$x = (2 + \cos\theta^1)\cos\theta^2; \quad y = \sin\theta^1; \quad z = (2 + \cos\theta^1)\sin\theta^2$$

for $\theta^1 \in [-\pi/2, 3\pi/2), \theta^2 \in [-\pi, \pi]$. Let $h \in C^\infty(T^2)$ be the "height function" given by $h(x, y, z) = z$ (restricted to the torus). Find the critical points of h, show that they are all nondegenerate and compute their indices. Sketch a picture of the torus, indicating the critical points. (Hint: the formulas describing the torus, when restricted to $\theta^1 \in (-\pi/2, 3\pi/2)$, $\theta^2 \in (-\pi, \pi)$, can be viewed as x^{-1} for a coordinate system (x, U) on T^2. All critical points on h lie in U. Use the basis $\left\{\frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2}\right\}$ to compute the Hessian matrix at each critical point: it is nothing but the matrix of second partial derivatives!)