EXAM DIFFERENTIAL MANIFOLDS, MARCH 19 2007, 9:00-12:00

READ THIS FIRST

- Put your name and student number on every sheet you hand in.
- You may do this exam either in English or in Dutch. Your grade will not only depend on the correctness of your answers, but also on your presentation; for this reason you are strongly advised to do the exam in your mother tongue if that possibility is open to you.
- Be clear and concise (and so avoid irrelevant discussions).
- Do not forget to turn this page: there are also problems on the other side.
- I will soon post a set of worked solutions on `http://www.math.uu.nl/people/looijeng/smoothman06.html`

(1) Let $\lambda \in \mathbb{C}$ have positive real part. Prove that the map $f : \mathbb{R} \to \mathbb{C}$ defined by $f(t) = e^{\lambda t}$ is an injective immersion whose image is not closed in $\mathbb{C}$. Is f an embedding?

(2) Show that real projective n-space P^n is orientable for n odd. Explain why P^n cannot be oriented when n is even.

(3) Let M be a manifold, $f : M \to \mathbb{R}^2$ a C^∞-map and put $N := f^{-1}(0,0)$. Let V and W be vector fields on M that lift ∂/∂_x resp. ∂/∂_y (so $D_pf(V_p) = \partial/\partial_x$ and $D_pf(W_p) = \partial/\partial_y$ for every $p \in M$).

  (a) Prove that N is a submanifold of M and that $[V, W]$ is tangent to it (i.e., restricts to a vector field on N).

  (b) Suppose that V and W generate flows on M (that we shall denote by H resp. I). Prove that the map $\mathbb{R}^2 \times N \to M$, $(a, b, p) \mapsto I_bH_a(p)$ is a diffeomorphism. (Hint: find a formula for its inverse.)

  (c) Prove that if V and W generate flows on M, then the inclusion $i : N \subset M$ induces an isomorphism on De Rham cohomology: $H^k(i) : H^k_{DR}(M) \to H^k_{DR}(N)$ is an isomorphism for all k.

(4) Let M be a compact manifold and denote by $\pi : S^1 \times M \to M$ the projection. A k-form α on $S^1 \times M$ can always be written

$$\alpha(\theta, p) = \alpha'(\theta, p) + d\theta \wedge \alpha''(\theta, p),$$

where α' and α'' are forms (of degree k resp. $k-1$) on M that depend on $\theta \in S^1$ and θ is the angular coordinate on S^1. Let $I(\alpha)$ be the $(k-1)$-form on M defined by $I(\alpha)(p) := \int_0^{2\pi} \alpha''(\theta, p) d\theta$.

(a) Prove that I commutes with the exterior derivative: $dI = Id$.

(b) Prove that I induces a linear map

$$I : H^k_{DR}(S^1 \times M) \to H^{k-1}_{DR}(M)$$

and show that this map is surjective.

(c) Prove that $H^k(\pi) : H^k_{DR}(M) \to H^k_{DR}(S^1 \times M)$ is injective and that its composition with I is zero.

(d) Prove that the image of $H^k(\pi)$ is the kernel of I. Conclude that $H^k_{DR}(S^1 \times M) \cong H^k_{DR}(M) \oplus H^{k-1}_{DR}(M)$.