SOLUTIONS OF THE DIFFERENTIAL MANIFOLDS EXAM, MARCH 19 2007

(1) *Let* $\lambda \in \mathbb{C}$ *have positive real part. Prove that the map* $f : \mathbb{R} \to \mathbb{C}$ *defined by* $f(t) = e^{\lambda t}$ *is an injective immersion whose image is not closed in* $\mathbb{C}$. *Is* f *an embedding?*
We have $|f(t| = e^{\mathrm{Re}(\lambda)t}$. So $|f|$ defines a diffeomorphism of $\mathbb{R}$ onto $(0, \infty)$ with inverse $\mathrm{Re}(\lambda)\log(t)$. This implies that f and its derivative are injective: f is an injective immersion. It is also a homeomorphism onto its image, for its inverse is the restriction of $z \in \mathbb{C} - \{0\} \mapsto \mathrm{Re}(\lambda)\log|z|$ to $f(\mathbb{R})$. This implies that f is an embedding.

(2) *Show that real projective* n*-space* P^n *is orientable for* n *odd. Explain why* P^n *cannot be oriented when* n *is even.*
We orient S^n as boundary of the unit ball: if we identify T_pS^n with the orthogonal complement of p in $\mathbb{R}^{n+1}$, then we stipulate that a basis $v_1, \dots, v_n$ of the latter is oriented if and only if the basis $(p, v_1, \dots, v_n)$ of $\mathbb{R}^{n+1}$ has positive determinant. The antipodal map, $\iota : S^n \to S^n$, is the restriction of $-1_{n+1} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and the latter has determinant $(-1)^{n+1}$. So the derivative of ι at p sends $(p, v_1, \dots, v_n)$ to $(-p, -v_1, \dots, -v_n)$. Hence $D_p\iota$ is orientation preserving if and only if n is odd.

We now think of P^n as obtained from the unit sphere $S^n \subset \mathbb{R}^{n+1}$ by identifying antipodal pairs. The corresponding map $f : S^n \to P^n$ is a local diffeomorphism, in particular the derivative of f at any $p \in S^n$ is an isomorphism and we have $D_pf = D_p(f\iota) = D_{-p}fD_p\iota : T_pS^n \to T_{f(p)}P^n$. We saw that for odd n, $D_p\iota$ is orientation preserving and so we may fix an orientation of $T_{f(p)}P^n$ by requiring that D_f (or $D_{-p}f$—this does not matter) is orientation preserving. This defines an orientation on P^n.

Let now n be even and positive. Suppose we have succeeded in orienting P^n. Since $D_p\iota$ is orientation reversing only one of D_pf and $D_{-p}f$ is orientation preserving. This singles out an element of the antipodal pair $\{p, -p\}$. We thus obtain a continuous section $g : P^n \to S^n$ of f. Since P^n is compact, so is its image $g(P^n)$. In particular, $g(P^n)$ is closed. Its complement is $-g(P^n)$ and hence also closed. So $\{g(P^n), -g(P^n)\}$ is a nontrivial splitting of S^n. This contradicts the fact that S^n is connected.

(3) Let M be a manifold, $f : M \to \mathbb{R}^2$ a C^∞-map and put $N := f^{-1}(0,0)$. Let V and W be vector fields on M that lift ∂/∂_x resp. ∂/∂_y (so $D_pf(V_p) = \partial/\partial_x$ and $D_pf(W_p) = \partial/\partial_y$ for every $p \in M$).
(3a) *Prove that* N *is a submanifold of* M *and that* $[V, W]$ *is tangent to it (i.e., restricts to a vector field on* N*).*
The assumption implies that for every $p \in M$, D_pf is a surjection. So f is a submersion and by our form of the implicit function theorem it then follows that $N = f^{-1}(0, 0)$ is a submanifold.

We give two proofs for the second part.
First proof: Choose at a point of N a coordinate chart $\kappa : U \to \mathbb{R}^m$ such that

that $\kappa_1 = f_1$ and $\kappa_2 = f_2$. So N is then given by $\kappa_1 = \kappa_2 = 0$. In terms of this chart V resp. W looks like

$$\partial/\partial x^1 + \sum_{i\geq 3} V^i(x)\partial/\partial x^i \quad \text{resp.} \quad \partial/\partial x^2 + \sum_{i\geq 3} W^i(x)\partial/\partial x^i.$$

An easy check shows that the Lie bracket of these two vector fields involves only terms of the form $\partial/\partial x^3, \dots, \partial/\partial x^m$ and so $[V, W]$ is tangent to N.
Second proof: Since V is a lift of $\partial/\partial x^1$ we have $V(f_1) = 1$ and $V(f_2) = 0$. Likewise $W(f_1) = 0, W(f_2) = 1$. So $[V, W](f_1) = VW(f_1) - WV(f_1) = -W(1) = 0$ and similarly $[V, W](f_2) = 0$. This means that $[V, W]$ is tangent to the fibers of f.
(3b) *Suppose that V and W generate flows on M (that we shall denote by H resp. I). Prove that the map $\mathbb{R}^2 \times N \to M$, $(a, b, p) \mapsto I_b H_a(p)$ is a diffeomorphism. (Hint: find a formula for its inverse.)*
We claim that $fH_t(p) - (t, 0)$ is constant equal to $f(p)$. For if we differentiate the lefthand side with respect to t, then we get $Df_{H_t(p)}(V_{H_t(p)}) - \partial/\partial x^1 = 0$. For a similar reason, $fI_t(p) - (0, t)$ is constant equal to $f(p)$. So if $p \in M$, and $r(p) := H_{-f_1(p)} I_{-f_2(p)} p$ then $fr(p) = f(p) - (f_1(p), 0) - (0, f_2(p)) = (0, 0)$. In other words, $r(p) \in N$. This defines a differentiable map $r : M \to N$. Then $(f_1, f_2, r) : M \to \mathbb{R}^2 \times N)$ is the inverse of the map $\mathbb{R}^2 \times N \to M$, $(a, b, p) \mapsto I_b H_a(p)$ and so the latter is a diffeomorphism.
(3c) *Prove that if V and W generate flows on M, then the inclusion $i : N \subset M$ induces an isomorphism on De Rham cohomology: $H^k(i) : H^k_{DR}(M) \to H^k_{DR}(N)$ is an isomorphism for all k.*
Under the above diffeomorphism, the inclusion i simply becomes the inclusion of N in $\mathbb{R}^2 \times N$ (as $\{(0, 0\} \times N$). We know that for any manifold N, the inclusion of N in $\mathbb{R} \times N$ (as $\{(0\} \times N$) induces an isomorphism on De Rham cohomology. Applying this twice (first to N, then to $\{(0\} \times N$) yields the result.

(4) Let M be a compact manifold and denote by $\pi : S^1 \times M \to M$ the projection. A k-form α on $S^1 \times M$ can always be written

$$\alpha(\theta, p) = \alpha'(\theta, p) + d\theta \wedge \alpha''(\theta, p),$$

where α' and α'' are forms (of degree k resp. $k - 1$) on M that depend on $\theta \in S^1$ and θ is the angular coordinate on S^1. Let $I(\alpha)$ be the $(k - 1)$-form on M defined by $I(\alpha)(p) := \int_0^{2\pi} \alpha''(\theta, p)d\theta$.
(4a) *Prove that I commutes with the exterior derivative: $dI = -Id$*[1].
We regard α' as a family of k-forms on M depending on the angular parameter θ. If $d_M\alpha'$ denotes the corresponding exterior derivative, then it is clear that

$$d\alpha' = d_M\alpha' + d\theta \wedge \frac{\partial \alpha'}{\partial \theta} \text{ and } d(d\theta \wedge \alpha'') = -d\theta \wedge d_M\alpha''.$$

[1]The formula to prove was erroneously stated as: $dI = Id$

It follows that $(d\alpha)'' = \partial\alpha'/\partial\theta - d_M\alpha''$. If we then integrate over θ we find that

$$(Id\alpha)(p) = \int_0^{2\pi} \frac{\partial\alpha'}{\partial\theta}(\theta,p)d\theta - \int_0^{2\pi}(d_M\alpha'')(\theta,p)d\theta =$$
$$= \alpha'(p,2\pi) - \alpha'(p,0) - \int_0^{2\pi}(d_M\alpha'')(\theta,p)d\theta =$$
$$= -d\Big(\int_0^{2\pi}\alpha''(\theta,p)d\theta\Big) = -dI(\alpha).$$

(4b) *Prove that I induces a linear map*

$$I : H^k_{DR}(S^1\times M) \to H^{k-1}_{DR}(M)$$

and show that this map is surjective.
It is clear that I is $\mathbb{R}$-linear. If α is closed, then so is $I(\alpha)$, for $dI(\alpha) = -Id(\alpha) = 0$. If α is exact, say $\alpha = d\tilde\alpha$, then $I(\alpha) = Id(\tilde\alpha) = -dI(\tilde\alpha)$ and hence $I(\alpha)$ is exact. So I induces a linear map as asserted. If β is a closed $(k-1)$-form on M, then $d\theta\wedge\beta$ is a closed k-form on $S^1\times M$ (for $d(d\theta\wedge\beta) = -d\theta\wedge d\beta = 0$) and we have $I(\beta) = 2\pi\beta$. So I maps (the class of) $(2\pi)^{-1}d\theta\wedge\beta$ to (the class of) β.
(4c) *Prove that $H^k(\pi) : H^k_{DR}(M) \to H^k_{DR}(S^1\times M)$ is injective and that its composition with I is zero.*
Let $i : M \to S^1\times M$ be the inclusion given by $p\mapsto(0,p)$. Then $\pi i : M\to M$ is the identity map and hence so is $H^k_{DR}(\pi i) = H^k_{DR}(i)H^k_{DR}(\pi)$. This implies that $H^k_{DR}(\pi)$ is injective. If β is a k-form on M, then $\pi^*_M\beta$ is the same k-form, but now thought of as a form on $S^1\times M$. In particular, $(\pi^*_M\beta)'' = 0$ and so $I\pi^*_M\beta = 0$. It follows that π^*_M maps to the kernel of $I : H^k_{DR}(S^1\times M) \to H^{k-1}_{DR}(M)$.
(4d) *Prove that the image of $H^k(\pi)$ is the kernel of I. Conclude that $H^k_{DR}(S^1\times M) \cong H^k_{DR}(M)\oplus H^{k-1}_{DR}(M)$.*
An element a of the kernel of $I : H^k_{DR}(S^1\times M) \to H^{k-1}_{DR}(M)$ is by definition represented by a closed k-form α on $S^1\times M$ with the property that $I(\alpha)$ is exact: $I(\alpha) = d\beta$ for some $(k-2)$-form β on M. We must show that it can be represented by the image of a closed k-form on M under π^*_M.

Consider the $(k-1)$-form $d\theta\wedge\beta$ on $S^1\times M$. Then $d(d\theta\wedge\beta) = -d\theta\wedge d\beta = -d\theta\wedge I(\alpha)$. So upon replacing α by $\alpha + d(d\theta\wedge\beta)$, we can always represent a by an α with $I(\alpha)=0$, that is, with $\int_0^{2\pi}\alpha''(t,p)dt = 0$ for all p. Then

$$\gamma(p,\theta) := \int_0^\theta \alpha''(t,p)dt.$$

is periodic in θ with period 2π and hence defines a $(k-1)$ form on $S^1\times M$. We have $(d\gamma)'' = -d\theta\wedge\alpha''$ and so if we replace α by $\alpha + d\gamma$, we can even arrange that $\alpha'' = 0$. We then have $\alpha = \alpha'$ and $0 = (d\alpha)'' = \frac{\partial\alpha'}{\partial\theta}$. This implies that α' is constant in θ and hence defines a k-form on M (that we

still denote α'). Then a is represented by $\pi_M^*(\alpha')$ and hence is in the image of $H^k_{DR}(\pi_M)$.
It follows from the preceding that

$$(H^k_{DR}(i), I) : H^k_{DR}(S^1 \times M) \to H^k_{DR}(M) \oplus H^{k-1}_{DR}(M)$$

is an isomorphism of vector spaces.