EXAM DIFFERENTIABLE MANIFOLDS, MARCH 16 2009, 9:00–12:00

READ THIS FIRST. *Be sure to put your name and student number on every sheet you hand in. And if a solution continues an another sheet, or if you want part of the submitted solution sheets to be ignored by the grader, then clearly indicate so.*

You may do this exam either in Dutch or in English, but whichever language you choose, be clear and concise. Books or notes (and neighbors for that matter) are not to be consulted.

Maps and manifolds are assumed to be of class $C^\infty$ unless stated otherwise.

A solution set will soon after the exam be linked at on the familiar Smooth Manifolds web page at `http://www.math.uu.nl/people/looijeng`

(1) Give an example of an injective immersion of between manifolds which fails to be an embedding.

(2) (a) Explain why any 2-form on a Möbius band must have a zero.
   (b) Let $M$ be a compact nonempty $m$-manifold and let $\omega$ be a nowhere zero $m$-form on $M$. Show that $M$ admits an orientation such that the integral of $\omega$ relative to this orientation is positive.

(3) Let $M$ be an $m$-manifold, $p \in M$ and $V$ a vector field on $M$ with $V_p \neq 0$. Let $H : (\varepsilon, \varepsilon) \times U \to M$ be a local flow of $V$, where $\varepsilon > 0$ and $U$ is a neighborhood of $p$. Let $N \subset U$ be a submanifold of $M$ of dimension $m - 1$ with $p \in N$ and $V_p \notin T_pN$.
   (a) Prove that the restriction of $D_pH$ to $\mathbb{R} \times T_pN$ (and going to $T_pM$) is an isomorphism of vector spaces.
   (b) Prove that $H$ maps a neighborhood of $(0, p)$ in $(\varepsilon, \varepsilon) \times N$ diffeomorphically onto a neighborhood of $p$ in $M$.
   (c) We use (b) to find a product neighborhood $(-\varepsilon', \varepsilon') \times N'$ of $(0, p)$ in $(\varepsilon, \varepsilon) \times N$ that is mapped by $H$ diffeomorphically onto a neighborhood $U'$ of $p$ in $M$ and denote by $G : U' \to (\varepsilon', \varepsilon') \times N'$ the inverse of this map. Prove that $G$ takes $V|U'$ to the vector field $(\frac{\partial}{\partial t}, 0)$.
   (d) Conclude that we can find a chart $(U''; \kappa^1, \dots, \kappa^m)$ of $M$ at $p$ on which $V$ takes the form $\frac{\partial}{\partial \kappa^1}$ and $N \cap U''$ is given by $\kappa^1 = 0$.

(4) Prove that any 1-from on the circle is uniquely written as the sum of an exact form and a constant multiple of $d\theta$, where $\theta$ is the angular coordinate (which, we recall, is only defined up to an integral multiple of $2\pi$).