Maps and manifolds are assumed to be of class C^∞ unless stated otherwise.

1. Give an example of an injective immersion of between manifolds which fails to be an embedding. Here are two examples: $f : t \in (-1,\infty) \mapsto (t^2-1, t^3-t)$ and $g : t \in \mathbb{R} \mapsto (e^{2\pi\sqrt{-1}t}, e^{a2\pi\sqrt{-1}t}) \in \mathbb{C}^2$ with $a \in \mathbb{R} - \mathbb{Q}$. In either case we have an injective immersion, but the map is not a homeomorphism onto its image, because the preimage of any neighborhood of $f(1)$ resp. $g(0)$ is never connected.

2. Explain why any 2-form on a Möbius band must have a zero. If a m-manifold M admits a nowhere vanishing m-form ω, then the charts (U,κ) for which $\omega|U$ has the form $\kappa^*(f dx^1 \wedge \cdots \wedge dx^m)$ with f a *positive* function on all of $\kappa(U)$, define an orientation for M. In particular, M is orientable. But we know that the Möbius band is not orientable.

Let M be a compact nonempty m-manifold and let ω be a nowhere zero m-form on M. Show that M admits an orientation such that the integral of ω relative to this orientation is positive. The above reasoning produces an orientation of M. We show this orientation is as desired. Since M is compact, we can cover M by a finite number of charts $(U_i,\kappa_i)_{i=1}^N$ such that $\omega|U_i = \kappa_i^*(f_i dx^1 \wedge \cdots \wedge dx^m)$, with f_i a positive on all of $\kappa_i(U_i)$. Choose a partition of unity $(\phi_i : M \to [0,1])_{i=1}^N$ subordinate to the covering $(U_i)_i$. We put $\psi_i := \phi_i\kappa_i^{-1} : \kappa_i(U_i) \to [0,1]$. Then ψ_i has compact support so that $\int_{\kappa_i(U_i)} \psi_i f_i dx^1 dx^2 \dots dx^m$ is a definite integral whose value is ≥ 0 (it is even > 0 unless the integrand is identically zero). This integral is by definition equal to $\int_M \phi_i\omega$; it is apparently > 0 unless $\phi_i\omega$ is identically zero. We see that $\int_M \omega = \sum_i \int_M \phi_i\omega$ is ≥ 0. If it were zero, then $\phi_i\omega = 0$ for all i and hence $\omega = 0$, which contradicts our assumption.

3. Let M be an m-manifold, $p \in M$ and V a vector field on M with $V_p \neq 0$. Let $H : (\varepsilon,\varepsilon) \times U \to M$ be a local flow of V, where $\varepsilon > 0$ and U is a neighborhood of p. Let $N \subset U$ be a submanifold of M of dimension $m-1$ with $p \in N$ and $V_p \notin T_pN$.

(a) Prove that the restriction of D_pH to $\mathbb{R}\times T_pN$ (and going to T_pM) is an isomorphism of vector spaces. Since $H(0,q) = q$ for all $q \in N$, the restriction of D_pH to $\{0\} \times T_pN \cong T_pN$ is simply the inclusion $T_pN \subset T_pM$. On the other hand, $t \mapsto H(t,p)$ has derivative in $t = 0$ equal to V_p, in other words, $D_pH(1,0) = V_p$. Since $V_p \notin T_pN$, it follows that the restriction of D_pH to $\mathbb{R} \times T_pN \to T_pM$ is injective. Since domain and range have the same dimension it must be an isomorphism of vector spaces.

(b) Prove that H maps a neighborhood of $(0,p)$ in $(\varepsilon,\varepsilon) \times N$ diffeomorphically onto a neighborhood of p in M. This follows from (a) and the inverse function theorem.

(c) We use (b) to find a product neighborhood $(-\varepsilon',\varepsilon') \times N'$ of $(0,p)$ in $(\varepsilon,\varepsilon) \times N$ that is mapped by H diffeomorphically onto a neighborhood U' of p in M and denote by $G : U' \to (\varepsilon',\varepsilon') \times N'$ the inverse of this map. Prove that G takes $V|U'$ to the vector field $(\frac{\partial}{\partial t}, 0)$. It is enough to show that the diffeomorphism $(-\varepsilon',\varepsilon') \times N' \to U'$ (a restriction of H) takes the the vector field $(\frac{\partial}{\partial t}, 0)$ to $V|U'$. But this is true by the very definition of a local flow.

(d) Conclude that we can find a chart $(U''; \kappa^1, \dots, \kappa^m)$ of M at p on which V takes the form $\frac{\partial}{\partial \kappa^1}$ and $N \cap U''$ is given by $\kappa^1 = 0$. Choose N' in (c) so small that it is the domain of a chart $(\lambda^1, \dots, \lambda^{n-1}) : N' \to \mathbb{R}^{m-1}$. Write $G = (G^1, G')$. Then $U'' := U'$ and $(\kappa^1, \cdots, \kappa^m) = (G^1, \lambda^1 G', \dots, \lambda^{m-1} G')$ is as desired.

4. Prove that any 1-form on the circle is uniquely written as the sum of an exact form and a constant multiple of $d\theta$, where θ is the angular coordinate (which, we recall, is only defined up to an integral multiple of 2π). Any 1-form ω on S^1 is written $f(\theta)d\theta$ with $f : \mathbb{R} \to \mathbb{R}$ periodic modulo 2π. Put $c := \frac{1}{2\pi}\int_0^{2\pi} f(t)dt$. Then the integral of $f - c$ over $[0, 2\pi]$ is zero. This means that $t \in \mathbb{R} \mapsto \int_0^t (f(\tau) - c)d\tau$ is periodic modulo 2π and hence defines a function $\phi : S^1 \to \mathbb{R}$. It is clear that $d\phi = \omega - cd\theta$. So $\omega = d\phi + cd\theta$ is written as an exact form plus a constant multiple of $d\theta$. To see that this way of writing is unique: if $\omega = d\phi' + c'd\theta$, then integration over the oriented circle yields

$$\int_{S^1} \omega = \int_{S^1} d\phi' + c' \int_{S^1} d\theta = 0 + 2\pi c',$$

from which it follows that $c' = c$.