EXAM DIFFERENTIABLE MANIFOLDS, 2008 JANUARY 28 9:00–12:00

You may do this exam either in Dutch or in English. Books or notes may not be consulted. Be sure to put your name on every sheet you hand in. Try to be clear and concise and if you want part of the submitted solution sheets to be ignored by the graders, then clearly indicate so.

Maps and manifolds are assumed to be of class C^∞ unless stated othewise. Numbers in brackets indicate the weight of the problem when we grade.

A solution set is available right after the exam (i.e., on Jan 28 2008 by noon) at
`http://www.math.uu.nl/people/looijeng/smoothman06.html`

(1) [5] Prove that an embedding of a manifold M in a manifold N followed by an embedding of N in a third manifold P is an embedding of M in P.

(2) [15] Let k and n be nonnegative integers and let $N_{k,n}$ be obtained from $(\mathbb{R}^k - \{0\}) \times \mathbb{R}^n$ by identifying (x, y) with $(-x, -y)$.
  (a) [5] Prove that $N_{k,n}$ is in a natural manner a manifold and that the projection $(\mathbb{R}^k - \{0\}) \times \mathbb{R}^n \to (\mathbb{R}^k - \{0\})$ induces a differentiable map $\pi : N_{k,n} \to N_{k,0}$.
  (b) [5] Prove that $N_{k,n}$ has in fact the structure of a vector bundle over $N_{k,0}$
  (c) [5] Prove that $N_{k,n}$ is orientable if $k + n$ is even.

(3) [10] Let M be a path-connected manifold and let α be a 1-form on M with the property that for every continuous, piecewise differentiable map $\delta : S^1 \to M$ we have $\int_{S^1} \delta^*\alpha = 0$.
  (a) [5] Prove that if α closed, then it is in fact exact. (Hint: integrate α along paths in M.)
  (b) [5] Prove that α is automatically closed. (Hint: prove this first in case M is an open subset of $\mathbb{R}^2$ and apply Stokes to a small disk in M.)

(4) [20] Let N be an oriented manifold of dimension $m + 1 \geq 1$ and $f : N \to \mathbb{R}$ a differentiable function whose differential df is nowhere zero.
  (a) [5] Prove that for every $t \in \mathbb{R}$, $N_{\leq t} := f^{-1}((-\infty, t])$ is a manifold with boundary $N_t := f^{-1}(t)$ and that N_t has a natural oriention.
  (b) [5] Let X be a vector field on N with the property that $X(f) = 1$. Prove that a local flow H of X satisfies $f(H(t, p)) = f(p) + t$.

  In the rest of this exercise we assume that for every $s \leq t$, $f^{-1}([s, t])$ is compact.

  (c) [5] Prove that for any closed m-form α on N, $\int_{N_t} \alpha$ is independent of t.
  (d) [5] In the following problem you may assume that X generates a flow $H : \mathbb{R} \times N \to N$ (although this actually follows from our data). Let μ be a $(m + 1)$-form on N with compact support. Prove that the function

$$F(t) := \int_{N_{\leq t}} \mu$$

  (where N_t is endowed with the orientation found in (a)) is differentiable and that its derivative in t equals $\int_{N_t} \iota_X(\mu)$.