SOLUTIONS DIFFERENTIABLE MANIFOLDS EXAM, 2008 JANUARY 28 9:00–12:00

(1) Prove that an embedding of a manifold M in a manifold N followed by an embedding of N in a third manifold P is an embedding of M in P.
A map f between differentiable manifolds is an embedding if and only if (i) f is an immersion and (ii) f maps the domain manifold homeomorphically onto its image in the target manifold. These two properties are preserved under composition. For instance, if f maps M homeorphically onto the subspace $f(M) \subset N$ and g maps N homemorphically onto the subspace $g(N) \subset P$, then gf maps M homemorphically onto the subspace $gf(M) \subset P$. This proves the asserted property.

(2) Let k and n be nonnegative integers and let $N_{k,n}$ be obtained from $(\mathbb{R}^k - \{0\}) \times \mathbb{R}^n$ by identifying (x, y) with $(-x, -y)$.
(a) Prove that $N_{k,n}$ is in a natural manner a manifold and that the projection $(\mathbb{R}^k - \{0\}) \times \mathbb{R}^n \to (\mathbb{R}^k - \{0\})$ induces a differentiable map $\pi : N_{k,n} \to N_{k,0}$.
(b) Prove that $N_{k,n}$ has in fact the structure of a vector bundle over $N_{k,0}$.
If $\tilde{U} \subset (\mathbb{R}^k - \{0\}) \times \mathbb{R}^n$ is such that $\tilde{U} \cap (-\tilde{U}) = \emptyset$, then the projection $(\mathbb{R}^k - \{0\}) \times \mathbb{R}^n \to N_{k,n}$ maps $\tilde{U}$ homeomorphically onto an open subset U of $N_{k,n}$. The inverse of that homeomorphism is then a chart for $N_{k,n}$. Two such charts with a common connected domain differ by a sign at most and so a coordinate change is differentiable. We further observe that $N_{k,n}$ is Hausdorff: two distinct points $p, q \in N_{k,n}$ have distinct representatives $\tilde{p}, \tilde{q} \in (\mathbb{R}^k - \{0\}) \times \mathbb{R}^n$ with $\tilde{q} \neq -\tilde{p}$. If $\tilde{U}_p \ni \tilde{p}$ and $\tilde{U}_q \ni \tilde{q}$ are disjoint neighborhoods with $\tilde{U}_p \cap (-\tilde{U}_q) = \emptyset$, then their images in $N_{k,n}$, $U_p \ni p$ and $U_q \ni q$ are disjoint as well.

The projection $\tilde{\pi} : (\mathbb{R}^k - \{0\}) \times \mathbb{R}^n \to (\mathbb{R}^k - \{0\})$ drops to a map $\pi : N_{k,n} \to N_{k,0}$. Let $\tilde{V} \subset \mathbb{R}^k - \{0\}$ be open and such that $\tilde{V} \cap (-\tilde{V}) = \emptyset$, so that $\tilde{V}$ defines an open subset V of $N_{k,0}$. Then $\tilde{V} \times \mathbb{R}^n$ defines the open subset $\pi^{-1}V$ of $N_{k,n}$. If we regard these a charts, then we see that π is differentiable.

The vector space structure on $\mathbb{R}^n$ turns π into a vector bundle: for any two pairs of charts as above, the transition function takes values in $\pm 1_n \in \mathrm{GL}(n, \mathbb{R})$.
(c) Prove that $N_{k,n}$ is orientable if $k + n$ is even.
We show that there is a natural orientation for every tangent space $T_pN_{k,n}$. Given $p \in N_{k,n}$, then let $\tilde{p}, -\tilde{p} \in (\mathbb{R}^k - \{0\}) \times \mathbb{R}^n$ be its preimages. Then the obvious isomorphism $T_{-\tilde{p}}\mathbb{R}^{n+k} \to T_pN_{k,n}$ is the composite of the derivative of minus the identity in $\mathbb{R}^{k+n}$ and the obvious isomorphism $T_{-\tilde{p}}\mathbb{R}^{n+k} \to T_pN_{k,n}$. Since $k + n$ is even, $-1_{k+n} \in \mathrm{GL}(\mathbb{R}, k + n)$ is orientation preserving and so either isomorphism defines the same orientation in $T_pN_{k,n}$

(3) Let M be a path-connected manifold and let α be a 1-form on M with the property that for every continuous, piecewise differentiable map $\delta : S^1 \to M$ we have $\int_{S^1} \delta^*\alpha = 0$.
(a) Prove that if α closed, then it is in fact exact.
Since α is closed, the Poincaré lemma implies that we can cover M with open subsets $U \subset M$ such that $\alpha|U = df_U$ for some $f_U : U \to \mathbb{R}$. If U is path connected and $p, q \in U$, then for any path $\gamma : [a, b] \to U$ from p to q we have

$$\int_\gamma \alpha = \int_a^b \gamma^* df_U = \int_a^b d(\gamma^* f_U) = f_U\gamma(b) - f_U\gamma(a) = f_U(q) - f_U(p).$$

This shows in particular that f_U is unique up to a constant.

Now fix $p_o \in M$. For every $p \in M$ we choose a differentiable path γ_p from p_o to p and put $f(p) := \int_{\gamma_p} \alpha$. We claim that $f(p)$ is independent of the choice of γ. For if γ_p' is another path from p_o to p, then traversing first γ_p and then γ_p' in reverse order defines a continuous, piecewise differentiable map $\delta : S^1 \to M$ with $\int_{S^1} \delta^* \alpha = \int_\gamma \alpha - \int_{\gamma'} \alpha$ and this is zero by assumption.

If U is a ball-like neighborhood of p , then we can define $f|U$ by means of paths that begin with γ_p and then stay in U. The above argument shows that $f|U$ is up to an additive constant equal to the f_U we found there. So $f|U$ is differentiable and $df|U = \alpha|U$.

(b) Prove that α is automatically closed.

In order to verify $d\alpha$ is zero in $p \in M$, choose a chart (U, κ) at p so that $\kappa(p) = 0$ and $\kappa(U)$ contains the unit ball. Write $\alpha = \sum_{1 \le i < j \le m} \kappa^*(a_{ij} dx_i \wedge dx_j)$, where $m = \dim M$. Now let for $1 \le i < j \le m$ and $\varepsilon < 1$, $D_{ij}(\varepsilon) \subset \mathbb{R}^m$ be the intersection of the ε-ball in $\mathbb{R}^m$ with the (x_i, x_j)-plane. Then $\kappa^{-1}(\partial D_{ij}(\varepsilon))$ is an oriented loop in M and so we have

$$
\begin{aligned}
0 = \int_{\kappa^{-1}(\partial D_{ij}(\varepsilon))} \alpha &= \int_{\kappa^{-1}(D_{ij}(\varepsilon))} d\alpha \text{ (by Stokes' theorem)} \\
&= \int_{D_{ij}(\varepsilon)} \sum_{k<l} a_{kl} dx_k \wedge dx_l) = \int_{D_{ij}(\varepsilon)} a_{ij} dx_i \wedge dx_j.
\end{aligned}
$$

If we divide the latter expression by $\pi\varepsilon^2$, then it tends to $a_{ij}(0)$ for $\varepsilon \to 0$. It follows that $a_{ij}(0) = 0$. So $d\alpha(p) = 0$.

(4) Let N be an oriented manifold of dimension $m + 1 \ge 1$ and $f : N \to \mathbb{R}$ a differentiable function whose differential df is nowhere zero.

(a) Prove that for every $t \in \mathbb{R}$, $N_{\le t} := f^{-1}((-\infty, t])$ is a manifold with boundary $N_t := f^{-1}(t)$ and that N_t has a natural oriention.

Let $p \in N_t$. By the implicit function theorem there exists a chart (U, κ) at p such that $\kappa_1 = f$. It follows that $N_{\le t} \cap U$ is defined by $\kappa_1 \le 0$. This shows that $N_{\le t}$ is a manifold with boundary. The orientation of N_t comes from the orientation of N and its description as a boundary: if we take κ to be oriented, then $(\kappa_2, \dots, \kappa_{m+1})$ is an oriented chart for N_t

(b) Let X be a vector field on N with the property that $X(f) = 1$. Prove that a local flow H of X satisfies $f(H(t, p)) = f(p) + t$.

It suffices to show that for a fixed p the derivative of $t \mapsto H(t, p)$ is constant equal to 1. This is indeed the case:

$$\frac{d}{dt} f(H(t,p)) = Df\Big(\frac{d}{dt} H(t,p)\Big) = Df(X_{H(t,p)}) = X(f)(H(t,p)) = 1.$$

In the rest of this exercise we assume that for every $s \le t$, $f^{-1}([s, t])$ is compact.

(c) Prove that for any closed m-form α on N, $\int_{N_t} \alpha$ is independent of t.

Let $s < t$. Then $f^{-1}([s, t])$ is a manifold with boundary. The boundary decomposes into $N_s \cup N_t$. The orientation it receives from $f^{-1}([s, t])$ is on N_t the one we found under (a) but on N_s is it opposite. So by Stokes' theorem

$$\int_{N_t} \alpha - \int_{N_s} \alpha = \int_{\partial f^{-1}([s,t])} \alpha = \int_{f^{-1}([s,t])} d\alpha = 0.$$

(d) In the following problem you may assume that X generates a flow $H : \mathbb{R} \times N \to N$ (although this actually follows from our data). Let μ be a $(m+1)$-form on N with compact

support. Prove that the function

$$F(t) := \int_{N_{\leq t}} \mu$$

(where N_t is endowed with the orientation found in (a)) is differentiable and that its derivative in t equals $\int_{N_t} \iota_X(\mu)$.

It follows from (b) that H_ε maps $N_{\leq t}$ onto $N_{\leq t+\varepsilon}$. So

$$F(t+\varepsilon) = \int_{H_\varepsilon(N_{\leq t})} \mu = \int_{N_{\leq t}} H_\varepsilon^* \mu$$

Differentiating this with respect to ε at $\varepsilon = 0$ yields

$$\frac{dF}{dt}(t) = \int_{N_{\leq t}} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} H_\varepsilon^* \mu = \int_{N_{\leq t}} \mathcal{L}_X \mu = \int_{N_{\leq t}} d\iota_X \mu = \int_{N_t} \iota_X \mu,$$

where in the last equality we applied Stokes' theorem.