Utrecht University
Mathematics
Stochastic processes
Fall 2011

Test, November 6, 2012

JUSTIFY YOUR ANSWERS

Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of five questions plus one bonus problem.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. Let X_i, $1 = 1, \ldots, n$ be independent normal random variables with respective means μ_i and variances σ_i^2. Consider its mean

$$\overline{X} \;=\; \frac{1}{n}\sum_{i=1}^{n} X_i$$

(a) (0.5 pts.) Prove that $\overline{X}$ is also normally distributed.

(b) (0.5 pts.) Determine the mean and variance of $\overline{X}$.

Answers: *The moment-generating function $\Phi_{\overline{X}}(t)$ of $\overline{X}$ factorizes, due to the independence of the X_i, in the following way:*

$$\Phi_{\overline{X}}(t) \;=\; E\big(\mathrm{e}^{t\overline{X}}\big) \;=\; \prod_{i=1}^{n} E\big(\mathrm{e}^{tX_i/n}\big) \;=\; \prod_{i=1}^{n} \Phi_{X_i}(t/n)$$

where Φ_{X_i} is the moment-generator function of the variable X_i. As each X_i is normal,

$$\Phi_{X_i}(s) \;=\; \exp\Big[\mu_i s + \frac{\sigma_i^2 s^2}{2}\Big]\;,$$

hence

$$\Phi_{\overline{X}}(t) \;=\; \exp\Big[t\Big(\frac{1}{n}\sum_{i=1}^{n}\mu_i\Big) + \frac{t^2}{2}\Big(\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2\Big)\Big]\;.$$

This is the moment-generating function of a normal variable with mean $\frac{1}{n}\sum_{i=1}^{n}\mu_i$ and variance $\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$. This identifies $\overline{X}$ as a variable with such a law.

Exercise 2. (1 pt.) Consider a branching process with offspring number with mean μ and variance σ. That means, a sequence of random variables $(X_n)_{n\geq 0}$ with $X_0 = 1$ and

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \qquad n \geq 1$$

where Z_n are iid random variables (offspring distribution) independent of the (X_n) with mean μ Show that $E(X_n) = \mu^n$. [*Hint:* Start by showing that $E(X_n) = \mu\, E(X_{n-1})$.]

Answer: *Start with*

$$E(X_n) \;=\; E\big(E(X_n \mid X_{n-1})\big)\;.$$

Now

$$\begin{aligned} E(X_n \mid X_{n-1} = x_{n-1}) &= E\Big(\sum_{n\geq 1}^{x_{n-1}} Z_i \,\Big|\, X_{n-1} = x_{n-1}\Big) \\ &= \sum_{n\geq 1}^{x_{n-1}} E(Z_i \mid X_{n-1} = x_{n-1}) \\ &= \sum_{n\geq 1}^{x_{n-1}} E(Z_i) \qquad \textit{(independence of } Z_i \textit{ and } X_{n-1}\textit{)} \\ &= x_{n-1}\,\mu\,. \end{aligned}$$

Hence $E(X_n \mid X_{n-1}) = \mu\, X_{n-1}$ *and*

$$E(X_n) = E\big(\mu\, X_{n-1}\big) = \mu\, E(X_{n-1})\,.$$

By induction in n *we get the proposed result.*

Exercise 3.

(a) (0.8 pts.) Show that

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}^n = \begin{pmatrix} 1/2 + a^n/2 & 1/2 - a^n/2 \\ 1/2 - a^n/2 & 1/2 + a^n/2 \end{pmatrix}$$

for $n \geq 1$. Determine a.

Answer: *This is an easy proof by induction. Comparing for the case* $n = 1$ *we obtain* $a = 2p - 1$.

(b) A communication system transmits the digits 0 and 1. Each digit must pass through n stages, each of which independently transmits the digit correctly with probability p.

-i- (0.8 pts.) Find the probability that the final digit, X_n, is correct.
Answer: $P^n_{00} = P^n_{11} = 1/2 - (2p-1)^n/2$.

-ii- (0.8 pts.) Find the probability that *all* the first n stages transmit correctly.
Answer: *By independence the probability is equal to* p^n.

Exercise 4. Consider a three-state Markov process $(X_n)_{n\geq 0}$ with two absorbing states. That is, a process with a three-symbol alphabet (=state space), say $\{0, 1, 2\}$, and transition matrix

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 \\ a & b & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c > 0$ and $a + b + c = 1$.

(a) (0.8 pts.) Show that $\mathbb{P}^n_{1\,1} = b^n$.

Answer: *As* $P_{x\,1} = 0$ *for every* $x \neq 1$,

$$P^n_{11} = \sum_{x=0}^{2} P^{n-1}_{1\,x} P_{x\,1} = P^{n-1}_{11} P_{11} = P^{n-1}_{11}\, b\,.$$

The result follows by induction.

(b) (0.8 pts.) Show that the state "1" is transient.

Answer:

$$\sum_{n\geq 0} P_{11}^n \;=\; \sum_{n\geq 0} b^n \;=\; \frac{1}{1-b} \;<\; \infty\,.$$

(c) (0.8 pts.) Let $T = \inf\{n > 0 : X_n = 0 \text{ or } X_n = 2\}$ be the time it takes the process to be absorbed in one of the absorbing states. Compute $E(T \mid X_0 = 1)$. [*Hint:* you may want to use that for a discrete random variable Z, $E[Z] = \sum_{k\geq 0} P(Z > k)$.]

Answer:

$$E(T \mid X_0 = 1) \;=\;$$
$$\sum_{k\geq 0} P(T > k \mid X_0 = 1) \;=\; \sum_{k\geq 0} P\big(X_n = 1, n = 1, \ldots, k \bigm| X_0 = 1\big) \;=\; \sum_{k\geq 0} b^k \;=\; \frac{1}{1-b}\,.$$

(d) (0.8 pts.) Let $T_0 = \inf\{n > 0 : X_n = 0\}$ and $T_2 = \inf\{n > 0 : X_n = 2\}$ be the absorption times at each of the absorbing states. Compute $P(T_0 < T_2 \mid X_0 = 1)$.

Answer:

$$P(T_0 < T_2 \mid X_0 = 1)$$
$$= \sum_{k\geq 0} P\big(X_{k+1} = 0, X_n = 1, n = 1, \ldots, k \bigm| X_0 = k\big) \;=\; \sum_{k\geq 0} P\big(X_{k+1} = 0 \bigm| X_k = 1\big)\, P_{11}^k$$
$$= \sum_{k\geq 0} a\, b^k \;=\; \frac{a}{1-b}\,.$$

(e) (0.8 pts.) Compute *all* the invariant measures of the process.

Answer: *Let* $\pi = (\pi_0, \pi_1, \pi_2)$ *be the invariant measure. The conditions* $\sum_x \Pi_x P_{x\,y} = \Pi_x$ *plus the normalisation condition* $\Pi_0 + \Pi_1 + \Pi_2 = 1$ *become:*

$$\begin{aligned} \Pi_0 + a\,\Pi_1 &= \Pi_0 \\ b\,\Pi_1 &= \Pi_1 \\ c\,\Pi_1 + \Pi_2 &= \Pi_2 \\ \Pi_0 + \Pi_1 + \Pi_2 &= 1\,. \end{aligned}$$

All their solutions are of the form $\Pi_1 = 0$*,* $\Pi_1 + \Pi_2 = 1$*. That is, the invariant measures* Π *take the form*

$$\Pi = (\lambda, 0, 1-\lambda) \;=\; \lambda\,(1,0,0) + (1-\lambda)(0,0,1) \quad \textit{for } 0 \leq \lambda \leq 1\,,$$

That is, the invariant measures are convex combinations of the measure concentrated in the state "0" and the measure concentrated in the state "2".

Exercise 5. At a certain beach resort a bad day is equally likely to be followed by a good or a bad day, while a good day is five times more likely to be followed by a good day than by a bad day. The number of interventions by lifesavers is Poisson distributed with mean 4 in good days and mean 1 in bad days. Find, in the long run,

(a) (0.8 pts.) The probability of the lifesavers not having any intervention in a given day.

(b) (0.8 pts.) The average number of interventions per day.

[Take $e^{-4} \sim 0.02$ and $e^{-1} \sim 0.4$.]

Answers: *Associating "good days" → 1 and "bad days" → 2, the weather pattern is a Markov process with transition matrix*

$$\begin{pmatrix} 5/6 & 1/6 \\ 1/2 & 1/2 \end{pmatrix} .$$

The proportion of good and bad days is determined, in the long run, by the invariant measure Π *of this chain. This measure satisfies:*

$$\left.\begin{array}{c} \frac{5}{6}\Pi_1 + \frac{1}{2}\Pi_2 = \Pi_1 \\ \Pi_1 + \Pi_2 = 1 \end{array}\right\} \quad \Longrightarrow \quad \Pi = \Big(\frac{3}{4}, \frac{1}{4}\Big) .$$

(a) *Let* S *be the number of interventions per day.*

$$\begin{aligned} P(S=0) &= P(S=0 \mid \textit{good day})\, P(\textit{good day}) + P(S=0 \mid \textit{bad day})\ P(\textit{bad day}) \\ &= e^{-4}\,\frac{3}{4} + e^{-1}\,\frac{1}{4} \ \sim\ 0.11 \end{aligned}$$

(b)

$$\begin{aligned} E(S) &= E(S \mid \textit{good day})\, P(\textit{good day}) + E(S \mid \textit{bad day})\, P(\textit{bad day}) \\ &= 4\cdot\frac{3}{4} + 1\cdot\frac{1}{4} \ =\ \frac{13}{4} \ =\ 3.25 . \end{aligned}$$

Bonus problem

Bonus Consider a homogeneous (or shift-invariant) Markov chain $(X_n)_{n\in\mathbb{N}}$ $(X_n)_{n\in\mathbb{N}}$ with *finite* state space S. Let us recall that the *hitting time* of a state y is

$$T_y \ =\ \min\big\{n \ge 1 : X_n = y\big\} .$$

(a) If $\ell \le n \in \mathbb{N}$, $x, y \in S$, prove the following

-i- (0.5 pts.)

$$P\big(X_n = y, T_y = \ell \bigm| X_0 = x\big) \ =\ P_{yy}^{n-\ell}\, P\big(T_y = \ell \bigm| X_0 = x\big) .$$

Answer: *Decomposing in terms of trajectories,*

$$\begin{aligned} &P\big(X_n = y, T_y = \ell \bigm| X_0 = x\big) \\ &\quad= \sum_{x_1,\ldots,x_{\ell-1}\ne y} P\big(X_n = y, X_\ell = y, X_{\ell-1} = x_{\ell-1}, \cdots, X_1 = x_1 \bigm| X_0 = x\big) \\ &\quad= \sum_{x_1,\ldots,x_{\ell-1}\ne y} P_{yy}^{n-\ell} P\big(X_\ell = y, X_{\ell-1} = x_{\ell-1}, \cdots, X_1 = x_1 \bigm| X_0 = x\big) \\ &\quad= \sum_{x_1,\ldots,x_{\ell-1}\ne y} P_{yy}^{n-\ell} P\big(T_y = \ell \bigm| X_0 = x\big) \end{aligned}$$

-ii- (0.5 pts.)

$$P_{xy}^n \ =\ \sum_{\ell=1}^{n} P_{yy}^{n-\ell}\, P\big(T_y = \ell \bigm| X_0 = x\big) .$$

Answer: *As*

$$\{X_n = y\} = \bigcup_{\ell=1}^{n} \{X_n = y, T_y = \ell\} ,$$

the union being disjoint, we conclude that

$$\sum_{\ell=1}^{n} P\big(X_n = y, T_y = \ell \bigm| X_0 = x\big) = P\big(X_n = y \bigm| X_0 = x\big) = P_{xy}^n .$$

The result follows, hence, by summing both sides of -i- with respect to ℓ.

(b) Conclude the following:

-i- (0.5 pts.) If every state is transient, then for every $x, y \in S$.

$$\sum_{n\geq 0} P_{xy}^n < \infty .$$

Answer: *By -ii- above,*

$$\sum_{n\geq 0} P_{xy}^n = \sum_{n\geq 0}\sum_{\ell=1}^{n} P_{yy}^{n-\ell}\, P\big(T_y = \ell \bigm| X_0 = x\big) .$$

Hence, interchanging the order of summation,

$$\begin{aligned}\sum_{n} P_{xy}^n &= \sum_{\ell\geq 1}\sum_{n\geq \ell} P_{yy}^{n-\ell}\, P\big(T_y = \ell \bigm| X_0 = x\big) \\ &= \sum_{\ell\geq 1} P\big(T_y = \ell \bigm| X_0 = x\big) \sum_{m\geq 0} P_{yy}^m \\ &= P\big(T_y < \infty \bigm| X_0 = x\big) \sum_{m\geq 0} P_{yy}^m .\end{aligned}$$

If y is transient, the last sum is finite.

-ii- (0.5 pts.) The previous result leads to a contradiction with the stochasticity property of the matrix $\mathbb{P}$. Hence not all states can be transient.

Answer: *Summing over y the inequality in (b)-i- we get*

$$\sum_{y}\sum_{n\geq 0} P_{xy}^n < \infty \tag{1}$$

(recall that S is finite). However, by stochasticity $\sum_y P_{xy}^n = 1$ for every $n \geq 0$. Hence,

$$\sum_{y}\sum_{n\geq 0} P_{xy}^n = \sum_{n\geq 0}\sum_{y} P_{xy}^n = \sum_{n\geq 0} 1 = \infty ,$$

in contradiction with (1).