JUSTIFY YOUR ANSWERS
Allowed: Calculator, material handed out in class, *handwritten* notes (*your handwriting*)
NOT ALLOWED: Books, printed or photocopied material

NOTE:

- The test consists of six problems plus one bonus problem

- The score is computed by adding all the credits up to a maximum of 10

Problem 1. Let $Z_1, Z_2, \ldots$ be independent random variables with the same mean μ. Let N be an non-negative integer-valued random variable independent from the previous ones. Consider the random variable

$$Y = \prod_{i=1}^{N} Z_i \ .$$

(a) (0.5 pts.) Prove that, if $\mu = 1$

$$E(Y) = 1 \ .$$

(b) (0.5 pts.) More generally, if $\phi(t)$ is the moment-generating function of N, prove that

$$E(Y) = \phi(\ln \mu) \ .$$

Problem 2. A dance floor is lighted with blue and red lights which are randomly lighted. The color of each flash depends on the color of the two precedent flashes in the following way:

-i- Two consecutive red flashes are followed by another red flash with probability 0.9.

-ii- A red flash preceded by a blue flash is followed by a red flash with probability 0.6.

-iii- A blue flash is followed by a red flash with probability 0.5 if preceded by a red flash, and with probability 0.4 if preceded by another blue flash.

(a) (0.5 pts.) Write a transition matrix representing the process.

(b) (0.6 pts.) Find, in the long run, the proportion of time the dance floor is under red light.

Problem 3. Two lamps are placed in an electric device. The mean lifetimes of these lamps are respectively 4 and 6 hours. One of the lamps has just burnt. Assuming that lifetimes are independent exponentially distributed random variables, compute

(a) (0.5 pts.) The probability that the burnt lamp be the one with larger mean lifetime.

(b) (0.6 pts.) The expected additional lifetime of the other lamp.

Problem 4. (1 pt.) Let $\{N(t), t \geq 0\}$ be a Poisson process with unit rate ($\lambda = 1$) and T a random variable independent of the process. Prove that

$$E\big[N(T^2)\big] - E\big[N(T)\big]^2 = \mathrm{Var}(T) \ .$$

Problem 5. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Let T_n denote the n-th inter-arrival time and S_n the arrival time of the n-th event. Let $t > 0$. Find:

(a) (0.5 pts.) $P\big(N(t) = 1, N(2t) = 2, N(3t) = 3\big)$.

(b) (0.5 pts.) $E\big[S_{10} \mid S_4 = 3\big]$.

(c) (0.8 pts.) $E\big[T_2 \mid T_1 < T_2 < T_3\big]$.

(d) (0.8 pts.) $E\big[N(t)\, N(2t)\big]$.

(e) (0.8 pts.) $E\big[N(t) \mid N(2t) = 5\big]$.

Problem 6. A service center consists of two servers, each working at an exponential rate μ. Customers arrive independently at a rate λ and wait in line till the first server becomes available. The system has a capacity of at most three customers.

(a) (0.6 pts.) Write the number of customers as a continuous-time Markov chain, that is, for $i = 0, 1, 2, 3$ determine the birth rates λ_i, death rates μ_i and transition probabilities P_{ij}.

(b) (0.6 pts.) If initially the process starts with no client present, determine the expected time needed to have three clients present.

(c) Determine the fraction of time in which:

   -i- (0.6 pts.) the service center is empty, and

   -ii- (0.6 pts.) the service center is full

<div align="center">

Bonus problem

</div>

Bonus 1. *(Invariant probabilities are indeed invariant!)* (1.5 pts.) Consider a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with countable state-space $S = \{x_1, x_2, \ldots\}$, waiting rates ν_i and embedded transition matrix P_{ij}, $i, j \geq 1$. Let $(P_i)_{i \geq 1}$ be an invariant probability distribution, that is, a family of positive numbers P_i satisfying $\sum_i P_i = 1$ and

$$\sum_{k:k \neq i} P_k\, \nu_k\, P_{ki} \;=\; \nu_i\, P_i$$

for all $i \geq 1$. Prove that if the process is initially distributed with the invariant law (P_i), this law is kept for the rest of the evolution. That is, prove that

$$P\big(X(0) = x_i\big) = P_i \quad \Longrightarrow \quad P\big(X(t) = x_i\big) = P_i$$

for all $t \geq 0$. *Suggestion:* Follow the following steps.

(i) Show that if $P\big(X(0) = x_i\big) = P_i$, then

$$P\big(X(t) = x_j\big) \;=\; \sum_i P_i\, P_{ij}(t) \,.$$

(ii) Use Kolmogorov *backward* equations to show that, as a consequence,

$$\frac{d}{dt} P\big(X(t) = x_j\big) \;=\; 0$$

for all $t \geq 0$.

(iii) Conclude.