Utrecht University
Stochastic processes
WISB362
Winter 2015

Exam March 14, 2015

JUSTIFY YOUR ANSWERS
Allowed: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED

NOTE:

- The test consists of six exercises for a total of 10 credits plus two bonus problems for a maximum of 1.5 pts.
- The score is computed by adding all the valid credits up to a maximum of 10.

Exercise 1. An individual traveling on the real line gives random steps with mean zero and variance proportional to the square of the distance to the origin. Explicitly, if X_i, $i = 0, 1, \ldots$ is his position at time i, then

$$E(X_n \mid X_{n-1}) = 0 \quad , \quad E(X_n^2 \mid X_{n-1}) = \alpha\, X_{n-1}^2 \, ,$$

with $\alpha > 0$. Find

(a) (0.5 pt.) The mean position $E(X_n)$.

(b) (0.5 pt.) The variance of the position, $\mathrm{Var}(X_n)$ in terms of α and $E(X_0^2)$.

Exercise 2. (0.5 pt.) Let X_i, $i = 1, 2, \ldots$ be a sequence of IID random variables with moment generating function $\phi_X(t)$. Let N be a Poisson random variable of mean λ, independent from the X_i, and let $S = \sum_{i=1}^{N} X_i$. Show that the moment-generating function of S is

$$E\big(\mathrm{e}^{tS}\big) \;=\; \mathrm{e}^{\lambda(\phi(t)-1)} \, .$$

Exercise 3. Two stars flare up independently at random. Each minute a non-flaring star has a 20% probability of flaring up. Once flaring, the start has a 40% probability of continuing flaring at the next minute. Let X_n be the number of stars flaring after n time units; it is a Markov process with state space $\{0, 1, 2\}$.

(a) (0.5 pt.) Find the transition matrix of the process X_n.

(b) (0.5 pt.) If both stars are flaring now, find the probability that they will both non-flaring in two minutes.

(c) (0.5 pt.) Find the proportion of time both stars are flaring simultaneously.

Exercise 4. Let X_1, X_2 and X_3 be independent exponential random variables with respective rates μ_1, μ_2 and μ_3. Find:

(a) (0.7 pt.) $P\big(X_1 < X_2 < X_3\big)$.

(b) (0.7 pt.) $E\big(X_2 \bigm| X_1 < X_2 < X_3\big)$.

Exercise 5. Let $\big\{N(t) : t \geq 0\big\}$ be a Poisson process with rate 2. Find

(a) (0.5 pt.) $P\big(N(2) = 1\,, N(10) = 4\,, N(15) = 7\big)$.

(b) (0.5 pt.) $E\big[N(17) \mid N(10) = 4\,, N(5) = 0\big]$.

(c) (0.6 pt.) $E\big[N(20) \mid N(17) = 7\big]$.

(d) (0.7 pt.) $E\big[N(17) \mid N(20) = 7\big]$.

Exercise 6. A pizza delivery has two identical scooters, one of which is kept as a backup. A scooter fails after an exponential time of rate λ, in which case it is sent to be repaired and replaced by the backup vehicle, if the latter is working. The repair service employs a technician that can repair only one machine at a time and takes an exponential time of rate μ to repair it. The repaired scooter becomes the new backup.

(a) (0.7 pt.) Model this process as a birth-and-death process with state $i=$ number of non-working scooters. Determine the birth and death rates.

(b) (0.7 pt.) Determine the expected time until both scooters are simultaneously in the service department.

(c) (0.7 pt.) Determine, in the long run, the proportion of time the shop has no working scooter and can not, therefore, accept deliveries.

(d) (0.7 pt.) Write the 9 backward Kolmogorov equations, and observe that they form three sets of three coupled linear differential equations.

(e) (0.5 pt.) If $\lambda = \mu$, prove that $P_{00}(t) - P_{20}(t) = \mathrm{e}^{-\lambda t}$.

Bonus problems

Only one of them may count for the grade

You can try both, but only the one with the highest grade will be considered

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Bonus 1. [Not all states can be transient] Consider a homogeneous (or shift-invariant) Markov chain $(X_n)_{n\in\mathbb{N}}$ $(X_n)_{n\in\mathbb{N}}$ with *finite* state space S. Let us recall that the *hitting time* of a state y is

$$T_y \;=\; \min\{n \ge 1 : X_n = y\}\;.$$

(a) If $\ell \le n \in \mathbb{N}$, $x, y \in S$, prove the following

-i- (0.5 pt.)

$$P\big(X_n = y, T_y = \ell \mid X_0 = x\big) \;=\; P_{yy}^{n-\ell}\, P\big(T_y = \ell \mid X_0 = x\big)\;.$$

-ii- (0.5 pt.)

$$P_{xy}^{n} \;=\; \sum_{\ell=1}^{n} P_{yy}^{n-\ell}\, P\big(T_y = \ell \mid X_0 = x\big)\;.$$

(b) Conclude the following:

-i- (0.3 pt.) If every state is transient, then for every $x, y \in S$.

$$\sum_{n\ge 0} P_{xy}^{n} \;<\; \infty\;.$$

-ii- (0.2 pt.) The previous result leads to a contradiction with the stochasticity property of the matrix $\mathbb{P}$. Hence not all states can be transient.

Bonus 2. [Invariant probabilities are indeed invariant] Consider a continuous-time Markov chain $\{X(t) : t \geq 0\}$ with countable state-space $S = \{x_1, x_2, \ldots\}$, waiting rates ν_i and embedded transition matrix P_{ij}, $i, j \geq 1$. Let $(P_i)_{i \geq 1}$ be an invariant probability distribution, that is, a family of positive numbers P_i satisfying $\sum_i P_i = 1$ and

$$\sum_{k:k \neq i} P_k \, \nu_k \, P_{ki} \;=\; \nu_i \, P_i$$

for all $i \geq 1$. Prove that if the process is initially distributed with the invariant law (P_i), this law is kept for the rest of the evolution. That is, prove that

$$P\big(X(0) = x_i\big) = P_i \quad \Longrightarrow \quad P\big(X(t) = x_i\big) = P_i$$

for all $t \geq 0$. *Suggestion:* Follow the following steps.

(i) (0.5 pt.) Show that if $P\big(X(0) = x_i\big) = P_i$, then

$$P\big(X(t) = x_j\big) \;=\; \sum_i P_i \, P_{ij}(t) \, .$$

(ii) (0.7 pt.) Use Kolmogorov *backward* equations to show that, as a consequence,

$$\frac{d}{dt} P\big(X(t) = x_j\big) \;=\; 0$$

for all $t \geq 0$.

(iii) (0.3 pt.) Conclude.