Utrecht University
Stochastic processes
WISB362
Winter 2015

Exam May 29, 2015

JUSTIFY YOUR ANSWERS
Allowed: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED

NOTE:

- The test consists of seven exercises for a total of 11 credits.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. Consider events A, B of a probability space. Prove the following:

(a) (0.5 pts.) If $\mathbb{P}(B) = 0$ then $\mathbb{P}(A \cup B) = \mathbb{P}(A)$.

(b) (0.5 pts.) If $\mathbb{P}(B) = 1$ then $\mathbb{P}(A \cap B) = \mathbb{P}(A)$.

Exercise 2. Let $X_1, X_2, \ldots,$ be independent identically distributed normal variables with mean μ and standard deviations σ^2. Let

$$S_n = \sum_{i=1}^{n} X_i \ .$$

(a) (0.6 pts.) Find the moment-generating function for S_n and identify the law of S_n.

(b) (0.8 pts.) Let N be a Poisson(λ) random variable. Compute the moment-generating function of S_N.

Exercise 3. Consider a random walk in $0, 1, \ldots, N$ with probability p of moving to the right, q of moving to the left and partial reflections at the endpoints. That is, a Markov process with transition probabilities

$$\begin{aligned}
P_{0,0} &= 1-p \\
P_{0,1} &= p \\
P_{i,i+1} &= p \qquad\qquad i = 1, \ldots, N-1 \\
P_{i,i} &= 1-p-q \quad i = 1, \ldots, N-1 \\
P_{i,i-1} &= q \qquad\qquad i = 1, \ldots, N-1 \\
P_{N,N-1} &= q \\
P_{N,N} &= 1-q
\end{aligned}$$

(a) (0.8 pts.) If initially the walker is at position 1, which is the probability that after 2 time units the walker is again in 1.

(b) (0.8 pts.) Find the reversible (hence invariant) probability for this process *for all* values of $p \in (0, 1)$.

Exercise 4. The ticket office of Leiden Central Station has two clerks whose service time are independent and exponentially distributed with respective rates λ_1 and λ_2. When customer C arrives, she finds both clerks busy, one with customer A and the other with customer B. He will be serviced by the first clerk that completes service to his current customer. Find:

(a) (0.8 pts.) The mean time needed for C to complete her ticket purchase.

(b) (0.8 pts.) The probability that C is *not* the last to leave the ticket office.

Exercise 5. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ. Find

(a) (0.8 pts.) $P\big(N(5) = 1\,, N(10) = 4\,, N(20) = 7\big)$.

(b) (0.8 pts.) $P\big(N(20) = 7 \mid N(5) = 1\,, N(10) = 4\big)$.

(c) (0.8 pts.) $P\big(N(5) = 1\,, N(10) = 4 \mid N(20) = 7\big)$.

Exercise 6. A piramidal molecule undergoes oscillations in which one of the atoms changes from un Up to a Bottom position, passing through a Centre position. Simulations show that the times the atom stays at each position are exponentially distributed with mean 2 (picoseconds) for the Up and Bottom positions and mean 1 for the Centre position. An atom in the Up or Bottom position always moves towards the Centre after leaving its position, while an atom in the Centre moves towards the bottom twice as often as towards the top.

(a) (0.8 pts.) Model this evolution as a continuous-time Markov chain among the positions T, C and B. That is, determine the abandoning rates ν_P, ν_C and ν_B and the transitions $\mathbb{P}_{ij}$ with $i, j = T, C, B$.

(b) (0.8 pts.) Determine, in the long run, the fraction of time spent by the atom in each of the three positions.

Exercise 7. Consider a pure death process with three states, that is, a process whose only non-zero rates are the death rates μ_1 and μ_2.

(a) (0.8 pts.) Write the five non-trivial forward Kolmogorov equations.

(b) (0.6 pts.) Find $P_{11}(t)$ and $P_{22}(t)$.