Utrecht University
Stochastic processes
WISB362
Winter 2016

Exam July 4, 2016

**JUSTIFY YOUR ANSWERS**
**Allowed: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED**

**NOTE:**

- The test consists of six exercises for a total of 12 credits.
- The score is computed by adding all the valid credits up to a maximum of 10.

**Exercise 1.** Prove the following:

(a) (0.5 pts.) If $X$ has an exponential distribution with rate $\lambda$ and $a > 0$, then $Y = aX$ has an exponential distribution of rate $\lambda/a$.

(b) (0.5 pts.) If $X_1, X_2, \ldots, X_k$ are independent random variables with Gamma distributions with parameters $(n_1, \lambda), (n_2, \lambda), \ldots, (n_k, \lambda)$, then their sum $Y = X_1 + X_2 + \cdots + X_k$ has a Gamma law with parameters $(n_1 + n_2 + \cdots + n_k, \lambda)$.

**Exercise 2.** Consider a branching process with offspring number with mean $\mu$ and variance $\sigma^2$. That means, a sequence of random variables $(X_n)_{n\geq 0}$ with $X_0 = 1$ and

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \qquad n \geq 1$$

where $Z_n$ are iid random variables (offspring distribution) independent of the $(X_n)$ with mean $\mu$ and variance $\sigma^2$.

(a) (1 pt.) Show that $E(X_n) = \mu^n$. [*Hint:* Start by showing that $E(X_n) = \mu\, E(X_{n-1})$.]

(b) (1 pt.) Show that the variances of the process satisfy the recursive equation

$$\mathrm{Var}(X_n) \;=\; \mu^{n-1}\sigma^2 + \mu^2\,\mathrm{Var}(X_{n-1})\;.$$

**Exercise 3.** (1pt.) Consider a Markov processes started in the invariant (or stationary) measure. If this measure is reversible, prove that the probability of visiting the states (letters) $x_1, x-2, \ldots, x_n$ in that order is equal to the probability of visiting them in the opposite order.

**Exercise 4.** Let $X_1$, $X_2$ and $X_3$ be independent exponential random variables with respective rates $\lambda_1$, $\lambda_2$ and $\lambda_3$. Compute:

(a) (0.7 pts.) $E\big(X_1 + X_2 \bigm| X_1 < X_2\big)$.

(b) (0.7 pts.) $E\big(X_2 \cdot X_3 \bigm| X_2 < X_3\big)$.

(c) (0.7 pts.) $E\big(X_2 \mid X_1 < X_2 < X_3\big)$.

**Problem 5.** Two clerks handle packages at a distribution center. Their processing times are independent and identically distributed, each following an exponential law of rate $\mu$. Packages are processed on a first-come first-serve basis as soon as a clerk becomes free.

(a) A package $P_3$ arrives and finds both clerks busy processing packages $P_1$ and $P_2$. Denote $W$ the waiting time of package $P_3$ until a clerk becomes free, $T_P$ its processing time once accepted by a clerk, and $T = W + T_P$ the total time elapsed between the arrival of the package $P_3$ and the completion of its processing.

-i- (1 pt.) Determine the law of $W$.

-ii- (1 pt.) Prove that $E(T) = 3/(2\mu)$.

(b) Packages arrive independently, exponentially at rate $\lambda$ and wait in line till the first clerck becomes available.

-i- (0.6 pts.) Write the number of packages present as a birth-and-death chain, that is, determine the birth rates $\lambda_n$ and death rates $\mu_n$.

-ii- (1 pt.) Determine the mean time needed for having three packages present.

-iii- (1 pt.) Determine the limiting probabilities $P_i$, $i \geq 0$. Under which condition do these probabilities exist?

-iv- (0.3 pts.) Show that if $\lambda = \mu$, in the long run there is at least one server idle 2/3 of the time.

**Exercise 6.** (1 pt.) Let $(\pi_i)_{0\leq i\leq n}$ be the invariant measure for the discrete-time Markov process on $S = \{0, 1, \ldots, n\}$ defined by a matrix $(P_{ij})_{0\leq i,j\leq n}$ with $P_{ii} = 0$. Prove that the measure

$$P_i \;=\; \frac{\pi_i/\nu_i}{\sum_j \pi_j/\nu_j} \qquad 0 \leq i \leq n$$

is then invariant for the continuous-time Markov chain with state space $S$, jump rates $\nu_i$ and transition probabilities $P_{ij}$, $0 \leq i, j \leq n$.