Stochastic Processes (WISB 362) - Final Exam

Sjoerd Dirksen

June 25, 2019, 13:30-16:30

Question 1 [4 points]

Recall that a random variable X with values in $\{0, \ldots, n\}$ has a binomial distribution with parameters (n, p), where $n \in \mathbb{N} \cup \{0\}$ and $0 \leq p \leq 1$, if

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, \ldots, n.$$

Compute the probability generating function of X. Use this to show that the sum of two independent, binomially distributed random variables with parameters (n_1, p) and (n_2, p) is binomially distributed with parameters $(n_1 + n_2, p)$.

Question 2 [14 points]

Consider a Markov chain with state space $\{0, 1, 2, 3, 4, 5, 6\}$ and transition probability matrix

$$P = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(a) (3 points) Draw a transition diagram, identify the communicating classes, and determine which classes are closed.

(b) (2 points) Determine for each state whether it is recurrent or transient.

(c) (2 points) Determine the period of each state.

(d) (7 points) Compute $\lim_{n\to\infty} p_{01}(n)$.

Hint: what happens during the first time step?

Question 3 [6 points]

Consider a Markov chain with state space I and let $J \subset I$. Consider the random times $(T_m)_{m\geq 0}$ defined by

$$T_0 = \inf\{n \geq 0 \ : \ X_n \in J\}$$

and for $m \geq 1$

$$T_m = \inf\{n > T_{m-1} \ : \ X_n \in J\}.$$

Suppose that $\mathbb{P}(T_m < \infty) = 1$ for all $m \geq 0$. Consider the stochastic process $(Y_m)_{m\geq 0}$ defined by $Y_m = X_{T_m}$. Show that this process is a Markov chain with state space J and transition probabilities $(q_{ij})_{i,j\in J}$, where

$$q_{ij} = \mathbb{P}_i(X_{T_1} = j), \qquad i, j \in I.$$

Question 4 [8 points]

Cars are passing by a gas station according to a Poisson process with rate λ per hour. Assume that each car, independently of all other cars, stops to refuel with probability p.

(a) (4 points) Let Y_t be the number of cars that has stopped at the gas station before time t. Show that $(Y_t)_{t\geq 0}$ is a Poisson process with rate λp.

(b) (4 points) Let $\lambda = 6$ and $p = \frac{1}{3}$. The gas station opens at 09:00. A company official visits the gas station at 09:30 and finds that the station has served three customers. The official will return to the station at 10:00 and at 11:00. What is the probability that the station will have served at least eight customers at 10:00 and exactly ten customers at 11:00?

Question 5 [4 points]

Determine for each of the statements below whether it is true or false and give a solid motivation for your answer.

(a) (2 points) If C is a closed communicating class of a Markov chain, then it is recurrent.

(b) (2 points) Let $(X_t)_{t\geq 0}$ be a birth process with rates $(q_j)_{j\geq 0}$. If the process explodes in a finite amount of time, then the rates q_j must grow at least as fast as e^{cj} for some constant $c > 0$, i.e., for some $c > 0$,

$$\lim_{j\to\infty} \frac{e^{cj}}{q_j} = 0.$$

Some important probability distributions

Discrete distributions

Name	Probability mass function	
Bernoulli(p)	$\mathbb{P}(X = a) = p = 1 - \mathbb{P}(X = b)$	
Binomial(n, p)	$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$,	$k = 0, \ldots, n$
Geometric(p)	$\mathbb{P}(X = k) = (1-p)^{k-1} p$,	$k \in \mathbb{N}$
Poisson(λ)	$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$,	$k \in \mathbb{N} \cup \{0\}$

Continuous distributions

Name	Probability density function
Uniform(a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$
Exponential(λ)	$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$
Normal(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x-\mu)^2/(2\sigma^2))$