Stochastic Processes (WISB 362) - Re-take Exam

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Question 1 [6 points]

Consider a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 1-p & p & 0 \\ q & 0 & 1-q \\ 1 & 0 & 0 \end{bmatrix},$$

where $0 \leq p \leq 1$ and $0 \leq q \leq 1$.

(a) (3 points) Determine the communicating classes of the Markov chain and determine which are closed.

(b) (3 points) Compute the mean recurrence time of the state 2.

Question 2 [8 points]

Consider a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ p & 1-p & 0 \end{bmatrix},$$

where $p = \frac{1}{16}$. Determine $p_{11}(n)$ for all $n \geq 1$.

Question 3 [6 points]

Let $X_1, \ldots, X_n$ be independent, exponentially distributed random variables with parameter λ. Use moment generating functions to show that $S_n = \sum_{i=1}^n X_i$ has a gamma distribution with parameters n and λ, i.e., its probability density function is given by

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$

Question 4 [8 points]

Suppose that customers arrive at a shop according to a Poisson process with rate λ per hour.

(a) (4 points) Compute the probability that the sixth customer arrives at least 5 hours later than the third customer.

Hint: Question 3 may be helpful.

(b) (4 points) The shop opens at 09:00. Suppose that we know that at most one customer has arrived by 10:00. Compute the probability that a total of 3 customers have arrived by 11:00 and a total of 5 customers have arrived by 13:00.

Question 5 [4 points]

A stochastic matrix P is called doubly stochastic if every column of the matrix sums to 1, i.e., $\sum_{i \in I} p_{ij} = 1$. Let $(X_n)_{n \geq 0}$ be an irreducible, aperiodic Markov chain with a doubly stochastic transition matrix and suppose that its state space consists of N states. Show that $\lim_{n \to \infty} p_{ij}(n) = \frac{1}{N}$ for all $i, j \in I$.

Question 6 [4 points]

Determine for each of the statements below whether it is true or false and give a solid motivation for your answer.

(a) (2 points) If every state of an irreducible Markov chain is null recurrent, then its state space is infinite.

(b) (2 points) Let $(X_n)_{n \geq 0}$ be a Markov chain and let A be a subset of its state space. Let L_A be the last time that $(X_n)_{n \geq 0}$ exits A, where we set $L_A = \infty$ if there is no last time that the process exits A. Then L_A is a stopping time.

Some important probability distributions

Discrete distributions

Name	Probability mass function	
Bernoulli(p)	$\mathbb{P}(X = a) = p = 1 - \mathbb{P}(X = b)$	
Binomial(n, p)	$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$,	$k = 0, \ldots, n$
Geometric(p)	$\mathbb{P}(X = k) = (1-p)^{k-1} p$,	$k \in \mathbb{N}$
Poisson(λ)	$\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$,	$k \in \mathbb{N} \cup \{0\}$

Continuous distributions

Name	Probability density function
Uniform(a, b)	$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$
Exponential(λ)	$f(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$
Normal(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-(x-\mu)^2/(2\sigma^2))$