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In 2006/2007, the course WISB362 was given by dr. A.V. Gnedin.

Stochastische Processen (WISB362)
4 July 2007

100 points in total. Provide detailed solutions!

Question 1 *(10 points)*

The behaviour of a slot machine can be described by a Markov chain with possible states $\{1, 2\}$. Each time a gambler inserts 3 cents in the machine for playing one game, the machine randomly changes the state according to the transition matrix

$$P = \begin{pmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{pmatrix}.$$

Then, the machine pays out 7 cents if the new state is 1, or pays nothing otherwise. Let R_n be the return of the gambler per one game after n consecutive games (in other words, R_n is the value obtained by computing the arithmetic average of wins/losses in n games). Find the limiting value $\lim_{n\to\infty} R_n$.

Question 2 *(15 points)*

Consider a Markov chain $(X_n, n \geq 0)$ with state-space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0.25 & 0 & 0.75 & 0 \\ 0 & 0.25 & 0 & 0.75 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \end{pmatrix}$$

a) Determine all stationary distributions. *(10 points)*

b) For $\tau_1 = \min\{n \geq 1 : X_n = 1\}$, compute $\mathbb{E}_1\tau_1$, the expected return time to state 1. *(5 points)*

Question 3 *(15 points)*

Find the stationary distribution for a random walk on $S = \{0, 1, 2, \ldots\}$ with transition probabilities

$$p_{j,j+1} = p,\ p_{j,j-1} = q \quad \text{for} \quad j \geq 1$$

$$p_{0,1} = p_{0,0} = 1/2,$$

where $0 < p < 1/2$, $p + q = 1$.

Question 4 *(15 points)*

A Markov chain with the state-space $S = \mathbb{Z}^2$ jumps from state $(i, j) \in S$ to each of the states $(i + 1, j), (i, j + 1), (i - 1, j), (i, j - 1)$ with the same probability $1/4$.

a) Find an invariant measure for this Markov chain. *(8 points)*

b) Find all invariant measures. *(7 points)*

Question 5 *(15 points)*

A speed camera on a highway detects vehicles travelling over the legal speed limit at times of a Poisson process, on the average 20 violators per hour. One violator was detected in time 13:00–13:05 and two violators in time 13:05–13:15.

a) What is the probability of this event? (Do not bother to give the numerical value unless you have a microcalculator with you.) *(5 points)*

Let $T_1 < T_2 < T_3$ be the times when the camera snapped the speeding cars.

b) Determine the joint density of (T_1, T_2, T_3). *(10 points)*

Question 6 *(20 points)*

Meteorites strike the Earth surface at times of a Poisson process of some rate λ. For a given time instant t, let T be the time of a strike most close to t.

a) What is the probability $\mathbb{P}(T > t)$? *(5 points)*

b) What is the distribution of $|T - t|$? *(5 points)*

c) Fix time 0 (for instance, noon 02.07.2007) and some $t > 0$. Let B be the time of the last meteorite strike before t, with the convention $B = 0$ is there are no strikes in the time from 0 to t; and let A be the time of the first strike after t. Determine the expected value $\mathbb{E}(A - B)$. *(10 points)*

Question 7 *(10 points)*

Let $\{X_1, X_2, \ldots, X_{N_1}\}$ be the random set of points of a Poisson process $(N_t, t \in [0,1])$ with rate λ (the points may be labelled by increase, so that $X_1 < X_2 < \ldots < X_{N_1}$). Note that this set is empty if $N_1 = 0$. Let $f : [0,1] \to [0,1]$ be the function $f(x) = 2x \pmod 1$. For instance, $f(0.3) = 0.6 \pmod 1 = 0.6$, $f(0.74) = 1.48 \pmod 1 = 0.48$. Show that $\{f(X_1), \ldots, f(X_{N_1})\}$ is again a collection of points of a Poisson process on the interval $[0,1]$ with rate λ.