Retake — Discrete Mathematics

17:00 - 19:30, July 2, 2024

- The exam contains four questions. Two of them you have seen before and two are new to you. You do **all of the four** questions. We will grade all of them and drop the question you did worst.
- Write only your **student ID** on your submission. (For unbiased grading.)
- Please be silent, move your phone into your bag and put it on silent mode, please refrain from using red ink or erasable markers, like pencils. You are allowed to take this sheet home with you.
- Students with **extra time** may stay until 20:00. All other students are allowed to leave whenever they are finished.

Flip this page at 17:00.

1. **Knowledge Question:** Let $P = (X, \prec)$ be a partially ordered set.

   a Define $\alpha = \alpha(P)$ and $\omega = \omega(P)$ (10).

   b Prove that $\alpha \cdot \omega \geq |X|$ (40).

2. **Knowledge Question:** This question is about planar graphs.

   a Give the definition of when a graph is maximal planar (5).

   b How many edges does a maximal planar graph on at least $n \geq 3$ vertices have (5)?

   c Prove the answer to the previous question (40).

3. **Surprise Question:** This surprise question is based on Knowledge Question 9. Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation. Given the set $S \subseteq \{1, \ldots, n\}$ and assume that $\pi(x) \in S$, for all $x \in S$. Then we can define the *induced permutation* $\pi_S = \tau$ as the permutation $\tau : S \to S$ with $\tau(i) = \pi(i)$.

   a Assume that there is an $\varnothing \neq S \subsetneq \{1, \ldots, n\}$ such that π_S is well-defined. Show that π is not a cycle. (5)

   b Let $\varnothing \neq S \subseteq \{1, \ldots, n\}$ be such $\pi(S) \subseteq S$ and inclusion minimal with this property. Let P be the collection of sets with this property. Show that any two sets in P are disjoint. (10)

   c Show that every element $x \in \{1, \ldots, n\}$ is in some set $S \in P$. (10)

   d Show that, for each set $S \in P$, π_S is a cycle (5).

   e Using the previous statements show that each permutation decomposes into cycles (5).

   f We say π has *cycle number* k if it decomposes into k cycles. Let $Q2$ be the set of permutations with cycle number 2 on n elements. Give a formula that describes the number of permutations of $Q2$. Argue why your formula is correct. Hint: The number of cyclic permutations on a set with m elements is $(m-1)!$ (15).

4. **Surprise Question:** This surprise question is based on Knowledge Question 24.
   You are given a directed Eulerian graph $G = (V, E)$.

   a Give the two characteristics of Eulerian directed graphs (5).

   b Suppose you are given another directed Eulerian graph $G' = (V, E')$. Show that $G \cup G' = (V, E \cup E')$ might not be Eulerian (10).

   c Now, suppose we do not know if $G \cup G'$ is Eulerian. Describe an algorithm that outputs a boolean value based on whether $G \cup G' = (V, E \cup E')$ is Eulerian. The algorithm must run in $O(|E| + |E'|)$ time (10).

   d Assume, we know that G is Eulerian, E is given by an adjacency matrix and E' is given as an adjacency list. Explain how your algorithm can be modified to run in $O(|E'|)$ time (15). (Hint: If an edge set E is given as an adjacency matrix, then we can determine of $ij \in E$ in $O(1)$ time. If an edge set E is given as an adjacency list, then we can report all the neighbors of a vertex in $O(d)$ time, where d is the degree of that vertex.)

   e Prove the correctness of your algorithm from **c** by showing it correctly verifies the two characteristics from **a** (10).

List of Knowledge Questions

1. Give the definition of the following terms discussed in the lecture: poset (5), Hasse diagram (5), immediate predecessor/cover relation (5). Let $(X, \prec)$ be a poset and $\lhd$ the immediate predecessor relation and assume that X is finite. Then $x \prec y$ if and only if $\exists x_1, \ldots, x_k : x \lhd x_1 \lhd \ldots \lhd x_k \lhd y$ (35).

2. Give the definition of the following terms discussed in the lecture: linear order (5), lexicographic order (on the set $X_1 \times \ldots \times X_k$) (5). Show the following: State and show the Erdős-Szekeres theorem as in the book (40).

3. Given a partial order, define smallest/minimum element (5), minimal element (5). Show that every finite poset has a minimal element (40).

4. Given a partial order, define linear extension (10). Show that every finite poset has a linear extension (40).

5. This exercise is about posets, denoted by $P = (X, \prec)$. Define an embedding in the context of posets. (5) Show that every finite poset $P = (X, \prec)$ has an embedding into $(2^X, \subseteq)$ (35). Define chains and anti-chains in the context of posets (10).

6. Define $\alpha = \alpha(P)$ (5) and $\omega = \omega(P)$ (5) and prove that $\alpha \cdot \omega \geq |X|$ (40).

7. Let $n = |X|$ and $m = |Y|$. How many functions are there from X to Y (5)? Prove your answer (30). Use the last question to show that there are 2^n subsets of X (15).

8. Let $n = |X|$. Show that there are 2^n subsets of X by induction (25). Show that the X has 2^{n-1} subsets of odd size (25).

9. Let $n = |X|$ and $m = |Y|$. How many *injective* functions are there from X to Y (5). Prove your answer (30). Define a permutation (5) and the so-called one-row notation(5). Define a cycle in a permutation (5).

10. Define factorial and binomial coefficients $\binom{n}{k}$ (10). Show that the number of subsets of size k of X equals $\binom{n}{k}$ (40).

11. Let $(i_1, \ldots, i_r) \in \mathbb{N}^r$ be integers adding to n. How many such tuples of integers are there (5)? Give a proof to your answer (30). Show that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ (15).

12. Describe Pascal's triangle (5). State the Binomial Theorem (10). Show that $\sum_{i=0,\ldots,n} \binom{n}{i}^2 = \binom{2n}{n}$ (30). Define Multinomials and use them to count the number of ways to rearrange the letters in the word "MISSISSIPPI" (5).

13. Define the Harmonic numbers H_n (5). Show that $1/2\lfloor \log n \rfloor \leq H_n \leq \log n + 1$ (25). Define the notation $f = O(g)$, for functions f, g (5). Let $f(n) = 1^3 + \ldots + n^3$. Show that $f(n) = O(n^4)$ (5). What is the "big-oh" notation useful for in general (10)?

14. What is the possible criticism of this notation (10)? Give a tight asymptotic estimate of the function $f(n) = 1^3 + \ldots + n^3$, both from above and below (10). Show that $n^{n/2} \leq n! \leq (\frac{n+1}{2})^n$ (30), as done in the lecture/book.

15. In the lecture, we gave a lower bound on the exponential function using a linear function. State this lower bound (5). Show that $n! \leq en \left(\frac{n}{e}\right)^n$, using integrals (30). (You can use basic facts from Analysis.) What is the Stirling formula? You don't have to memorize the formula, only what it does (5). Show that $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq n^k$, using the formula for the binomial (10).

16. Show (25) that $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$. Show (25) that $\frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor} \leq 2^n$.

17. Show (40) that $\frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}$. State the prime number theorem (10).

18. State the Inclusion-Exclusion principle for general n (10). Prove the Inclusion-Exclusion principle (40).

Midterm ends here.
Endterm stars here.

19. Give the definition of a subgraph (5). Give the definition of the term connectedness in the context of graphs (5). Give the definition of a walk in a graph (5). Give the definition of the connected components of a graph (5). Show that a graph is connected if and only if it has exactly one component (10). Give the definition of the distance function of a graph and show that it is a metric using the three defining properties (20).

20. Give the definition of an induced subgraph (5) and the adjacency matrix of a graph (5). Let A be the adjacency of the graph G and $B = A^k$ be the k-th power of A. Which information does B_{ij} convey combinatorially? (5) Give a proof of your answer (25). Describe a multi-graph in some way (10).

21. What is the score of a graph (5)? Show for a graph $G = (V, E)$ that $\sum_{v \in V} \deg(v) = 2|E|$ (5). Show that a graph is Eulerian if and only if it is connected and every vertex has an even degree (40).

22. Give the definition of the degree of a vertex (5). State the Handshake lemma (5). State and prove the score theorem (35). Give the definition of what it means for a graph to be Eulerian (5).

23. Give the definition of a directed graph (5). Give the definition of a directed tour of a graph (5). Give the definition of a directed Eulerian graph (5). Give the two characteristics of Eulerian directed graphs (5). Give the definition of the graph operations: edge deletion, edge addition, edge-subdivision, vertex deletion (10). Show that a graph is 2-connected if and only if for any two vertices exists a cycle containing them (20).

24. Give the definition of the indegree, outdegree (5) and symmetrization for a directed graph (5). Give the definition of what it means for a graph to be k-vertex-connected and k-edge-connected (5). Assume we have a wheel with n 0/1's on the boundary. We can see k consecutive of them through an opening at the top. Given those k numbers we know the exact rotation of the wheel. What is the largest possible n (5)? Proof your answer (30).

25. Show that G is 2-connected if and only if any subdivision of G is 2-connected. (20) Show that every 2-connected graph can be generated from a triangle by edge additions and subdivisions (30).

26. Let $T(n)$ denote the maximum number of edges a graph on n vertices without a triangle can have. Show that $T(n) = \lfloor \frac{n^2}{4} \rfloor$ (40). Give the definition of what it means for a graph to be extremal (10).

27. Give all five **(i) - (v)** defining properties of trees that we had in the book (15). Show the equivalence of any three of the five defining properties (35). (You probably need to prove two lemmas as preparation.)

28. Give the definition of a rooted tree (5). Give the definition of the parent/child relationship (5). Give the definition of a planted tree (5). Give the definition of the code of a planted tree as in the book (5). Show that there is a injective mapping from planted trees to codes (10). How can we use the code for planted tree isomorphism testing (10)? How can we use the codes for planted trees to assign rooted trees codes that preserve isomorphism (10).

29. Give the definition of the eccentricity of a vertex (5). Give the definition of the center vertices of a graph (5). Show that every tree has either one or two center vertices (30). How can we use the center vertices and codes for rooted trees to assign unrooted trees codes that preserve isomorphism (10).

30. For this question, $G = (V, E)$ denotes a graph and $n = |V|$, and $m = |E|$. Give the definition of a spanning tree (5). Give the definition of a forest (5).
Explain Kruskal's algorithm to compute a spanning tree in an unweighted graph G (10). Show: If the algorithm returns a graph with $n - 1$ edges then it is a spanning tree and if the algorithm returns a graph F on $n - k$ edges then it is a forest with k connected components. Furthermore, show that the components of G and F are exactly the same. (30). (**Hint:** You may assume here that every cycle-free graph on $m = n - 1$ edges is a tree. You may also use the fact that a forest with $n - k$ edges has k components.)

31. Describe the UNION-FIND problem and describe one possible solution to it (20). Show that your solution can do $n - 1$ Union and m Find operations in $O(n \log n + m)$ time (20). How can a UNION-FIND data structure be used for a spanning tree algorithm and why does it use at most $n - 1$ Union and $2m$ Find operations? (10)

32. Describe Jarník's algorithm (ignoring the edge weights) to compute a spanning tree in an unweighted graph (15). Let T be the graph computed by the algorithm. Show that if $|V(T)| = n$ then T is a spanning tree (10). Show that if $|V(T)| < n$ then T is a spanning tree of one component of G (10). Give the definition of a weighted graph and minimum weight spanning trees (5). Describe Kruskal's algorithm completely (10).

33. Given a connected weighted graph G used as input, prove that Kruskal's algorithm returns a *minimum* spanning tree (40). (Here, you can assume that you already know that it will return a spanning tree.)
Describe the idea of greedy algorithms in general (10).

34. We will give a weighted surprise graph in the exam with a marked start vertex. Give the trees computed by Jarník's algorithm after each step (10). Describe Jarník's algorithm on weighted graphs (10). Show the correctness of Jarník's algorithm (30).

35. Prove that K_5 is not planar, without using Euler's formula (20). Prove that if G is a 2-vertex-connected planar graph, then every face in any planar drawing of G is a region of some cycle of G (20). State Kuratowski's theorem (10).

36. Give the mathematical definition of the notion of a planar drawing of a graph (10) and a face in a planar drawing (5). Show how to manufacture the Möbius band (5), the torus (5), and the Klein bottle (5) from a rectangle. Prove that any finite graph can be drawn without edge crossings on a "sphere with sufficiently many handles" (bounded surfaces with sufficiently high genus is the formally correct term.) (20).

37. Prove that a graph can be drawn on the sphere if and only if it can be drawn in the plane (20). Draw $K_{3,3}$ on the Möbius band without edge crossings (5), K_5 on the torus without edge crossings (5), and $K_{4,4}$ on the torus without edge crossings (10). State the Jordan curve theorem (10).

38. State Euler's formula and under which conditions it holds (10). Proof of the formula (30). State one example where Euler's formula can be applied (10).

39. Give the definition of regular polytopes in $\mathbb{R}^3$ (15). Show that there are at most five regular polytopes in $\mathbb{R}^3$ (35).

40. Give the definition of when a graph is maximal planar (5). How many edges does a maximal planar graph on at least $n \geq 3$ vertices have (5)? Proof the answer to the previous question (40).

41. Give the definition of the average and the minimum degree of a graph (5). (They are what you think they are.) Give an upper bound on the average and the minimum degree of a planar graph (10). (You are allowed to use Proposition 6.3.3 (i).) Give the definition of the chromatic number of a graph (10). During the exam, we will give you a "surprise" drawing D of a planar graph F. Copy that drawing D and draw the corresponding dual graph F^* in a different colored pen (10). Draw a graph G such that F is a subgraph of G^* (15).

42. Give the definition of edge contractions (5). And sketch an argument, for why edge contractions preserve planarity (10). Show the five-color theorem for planar graphs (35).