University of Utrecht
Introductory mathematics for finance
WISB373
Winter 2014

Problem set 1: Replication, binary models

I: Risk-free assets
(Reference: M. Capiński and T. Zastawniak: *Mathematics for finance*, Springer, 2003)

Exercise 1.

(a) An *annuity* is a sequence of yearly payments of an amount C for a number n of years. If the yearly (simple or effective) interest rate is r, prove that the fair purchase value (=present or discounted value of all the payments) is $A(n,r)\,C$ where

$$A(n,r) = \frac{1-(1+r)^{-n}}{r}.$$

(b) Likewise, a *perpetuity* is an infinite sequence of yearly payments of an amount C. Deduce the fair purchase value of a perpetuity if the yearly (simple or effective) interest rate is r.

Exercise 2. You subscribe a loan for P euros to be payed in n monthly instalments of C euros each at an effective interest r per year. Prove the following:

(a) The monthly payment is

$$C = \frac{P}{A(n,r/12)}$$

where $A(n,\tilde{r})$ was defined in the previous exercise.

(b) After k payments the outstanding amount —that is, the amount you have to pay to cancel the loan— is

$$P\,\frac{\left(1+\frac{r}{12}\right)^n - \left(1+\frac{r}{12}\right)^k}{\left(1+\frac{r}{12}\right)^n - 1}.$$

(c) The interest paid in the k-th payment is

$$P\,\frac{r}{12}\,\frac{\left(1+\frac{r}{12}\right)^n - \left(1+\frac{r}{12}\right)^{k-1}}{\left(1+\frac{r}{12}\right)^n - 1}.$$

(d) The part of the principal repaid in the k-th payment is

$$P\,\frac{r}{12}\,\frac{\left(1+\frac{r}{12}\right)^{k-1}}{\left(1+\frac{r}{12}\right)^n - 1}.$$

(e) The limit $n \to \infty$ gives a payment schedule in which you are only paying the interest without ever repaying the principal (a popular scheme for home sales in The Netherlands).

Exercise 3.

(a) What is the maximal mortgage you can subscribe if you can afford to pay 1000 E/month, according to the following plans:

-i- 10-year mortgage at 4% annual rate.

-ii- 20-year mortgage at 5% annual rate.

(b) You subscribe the second plan for the maximal amount, but after five years you get a new job and ask the bank to readjust the instalments to 1200 E/month. Find how much you shorten the repayment time.

Exercise 4. Two simple exercises to appreciate the effect of compounding:

(a) For your retirement you deposit 100 E/month for 40 years, subject to an annual compounding interest of 5%. Find the amount collected at the end.

(b) In 1692 Peter Minuit, governor of the colony of New Netherland, bought the island of Manhattan from indians for goods (beads, cloth, etc) worth 20 E. Find the value of this sum in 2013 at 5% compound

-i- continuously,

-ii- annually.

Exercise 5. Two simple questions to relativize the effect of different compounding schemes:

(a) Find the rate for continuous compounding equivalent to monthly compounding.

(b) Find the frequency of periodic compounding at 20% to be equivalent to annual compounding at 21%

Exercise 6. Find the price of a bond with face value 100E and 5E annual coupons maturing in four years, given that the continuous compounding rate is

(a) 5%,

(b) 8%.

Part II: No arbitrage. Replication

Exercise 7. The exchange rate from pound to euro is 1.6 euro per pound. A speculator foresees that in one year there is a 50% probability that the rate grow to 2 euros/pound and a 50% probability that it decrease to 1.4 euros/pound. Assume that safe bonds pay no interest. He decides to buy a put option to sell 100 pounds at 1.8 euros/pound and he pays 10 euros for the option. Determine wether this is the fair price of the option and, if not, describe an arbitrage strategy.

Exercise 8. [Exchange rate] Let u be the continuously-compound borrowing rate in the UK and r that in continental Europe. An investor wants to fix a exchange rate V_T in a forward contract in which V_T euros are exchanged for one pound at time T. What is the fair value of C_T if the current rate of the pound is C_0 euros?

Exercise 9. No arbitrage for binary stock prices

(a) Show that the binary evolutions of stock prices offers no arbitrage opportunity if, and only if,

$$S_1(T) \, < \, S_0 \, G_0 \, < \, S_1(H) \, . \tag{1}$$

(b) Show that a forward contract offers no arbitrage opportunity if, and only if,

$$K \, = \, S_0 \, G_0 \, . \tag{2}$$

Exercise 10. [No arbitrage in a binomial market]

(a) Do Exercise 1.2 of Section 1.6 (Page 20) of Shreve's book. [*Hint:* Prove that $X_1(H) + X_1(T) = 0$.]

(b) Do its general version: If the option has value V_1 and a general binary tree, the portfolio of the investor after one period is worth

$$X_1 = \Delta_0 S_1 + \Gamma_0 V_1 - G_0 (\Delta_0 S_0 + V_0 \Gamma_0) .$$

Show that no arbitrage is possible for the investor if V_0 is the fair price of the option. [*Hint:* use condition (1) or, alternatively, prove that $\widetilde{p} X_1(H) + \widetilde{q} X_1(T) = 0$.]

Exercise 11. [Hedging for the option's buyer]

(a) Do Exercise 1.6 of Section 1.6 (Page 20) of Shreve's book.

(b) More generally, show that an option buyer can hedge his position in one period (that is, guarantee that no money will be lost whether the market comes up "H" or "T") by doing exactly the opposite of the option's seller (that is, short-selling stock when the seller buys and vice-versa).

Exercise 12. [Replication with selling fee] Two scenarios are foreseen for a certain stock after one period: one in which the stock value is 110 E and another in which the value is 90E. Its current value is $S_0 = 100$E. Furthermore, each operation of selling the stock to the market carries a fee of 2% (there is no fee to buy from the market). A call option is established at a strike price also equal to 100E. Determine

(a) The risk-neutral probability.

(b) The fair price of the option.

(c) The hedging strategy.

Exercise 13. [Replication with interest spread] Consider a stock whose present value is 75E and whose value at the end of one period can either grow to 90 or decrease to 67.5. An investor wants to buy an option for 75E at the end of the period in a market where borrowing money costs 12% and deposits pay only 8%.

(a) For a call option, determine:

-i- The risk-neutral probability.

-ii- The fair price of the option.

-iii- The hedging strategy.

(b) Repeat the calculations for a put option.

Exercise 14. ["Lookback" and Asian options] A stock evolves following a binomial model with $u = 2$ y $d = 1/2$. Consider the following option schemes for N periods:

(a) *"Lookback" option:* This is an exotic option whose final payoff is the difference between the highest historic stock price and the final stock price:

$$V_N(\omega) = \max_{0 \leq n \leq N} S_n(\omega) - S_N(\omega) .$$

(b) *Asian option:* The final payoff depends on the average price of the stock in the N periods:

$$V_N(\omega) = \left| \frac{1}{N+1} \sum_{n=0}^{N} S_n(\omega) - K \right|_+ .$$

If the options are established for $N = 3$ periods, $S_0 = K = 4$ and the interest rate for each period is 5%, determine in each case:

-i- The risk-neutral probability.

-ii- The fair price of the option.

-iii- The hedging strategy.

Exercise 15. [Digital option] This is an option in which the final payoff depends discontinuously on the final stock price. A particular case is the *cash-or-nothing* option which can be thought as a bet on the final asset price: the buyer receives a pre-specified amount X if this price exceeds a closing value K at a final time T, and otherwise receives nothing. Find the fair price of the option, for $T = N$ periods of a market evolving as a binomial model [$S_{n+1}(H) = uS_n$, $S_{n+1}(T) = dS_n$].

Exercise 16. [Binary additive tree] Consider an asset whose price grows or decreases a fixed amount in each period. That is, its price evolves according to a binary tree with

$$S_{n+1} = S_n + U \qquad S_{n+1} = S_n - D \ .$$

Assuming that the option is established for N periods for a strike value equal to the present value, $S_0 = K$, determine:

(a) The risk-neutral probability.

(b) The different stock values S_n

(c) The different prices V_n of the security: in particular the fair price of the option.

(d) The hedging strategy.