Utrecht University
Introductory mathematics for finance
WISB373
Winter 2015

Exam March 13, 2015

JUSTIFY YOUR ANSWERS
Allowed material: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED

NOTE:

- The test consists of four questions for a total of 10 points plus a bonus problem worth 1.2 points.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Coupon bond] Consider a coupon bond with face value F and maturity equal to N years, paying a coupon C at the end of each year. The interest rate is r per year, continuously compounded.

(a) (0.5 pt.) Show that the price of such bond is

$$V \;=\; C\frac{1-\mathrm{e}^{-rN}}{\mathrm{e}^{r}-1} + F\mathrm{e}^{-rN}\;.$$

(b) (0.5 pt.) An investor purchases the bond but decides to sell it immediately after having received the k-th coupon. Find the selling price.

Answers:

(a)

$$\begin{aligned} V &= \sum_{n=1}^{N} C\mathrm{e}^{-rn} + F\mathrm{e}^{-rN} \\ &= C\mathrm{e}^{-r}\sum_{n=0}^{N-1} C\mathrm{e}^{-rn} + F\mathrm{e}^{-rN} \\ &= C\mathrm{e}^{-r}\frac{1-\mathrm{e}-rN}{1-\mathrm{e}^{-r}} + F\mathrm{e}^{-rN} \\ &= C\frac{1-\mathrm{e}^{-rN}}{\mathrm{e}^{r}-1} + F\mathrm{e}^{-rN}\;. \end{aligned}$$

(b) The argument is as above but involving only the $N-k$ remaining years, hence

$$V_k \;=\; C\frac{1-\mathrm{e}^{-r(N-k)}}{\mathrm{e}^{r}-1} + F\mathrm{e}^{-r(N-k)}\;.$$

Exercise 2. [Martingales super- and sub-martingales] An biased coin having probability p of showing heads is repeatedly tossed. Let $(\mathcal{F}_n)$ be the filtration of the binary model, in which $\mathcal{F}_n$ are the events determined by the first n tosses. A stochastic process (X_j) is defined such that

$$X_j \;=\; \begin{cases} 2 & \text{if } j\text{-th toss results in head} \\ -1 & \text{if } j\text{-th toss results in tail} \end{cases} \qquad \text{for } j=1,2,\ldots$$

In turns, this process defines a “random walk"

$$\begin{aligned} Y_0 &= 0 \\ Y_n &= \sum_{j=1}^{n} X_j \quad j\geq 1\;. \end{aligned}$$

(a) Determine for what values of p the process $(Y_n)_{n\geq 0}$ is

-i- (0.5 pt.) a martingale adapted to the filtration $(\mathcal{F}_n)$,

-ii- (0.5 pt.) a sub-martingale adapted to the filtration $(\mathcal{F}_n)$,

-iii- (0.5 pt.) a super-martingale adapted to the filtration $(\mathcal{F}_n)$.

(b) (0.7 pt.) Show that the process $(Y_n)_{n\geq 0}$ is Markovian for all values of p.

(c) (0.5 pt.) Prove that the process $(Y_n^4)_{n\geq 0}$ is a sub-martingale if so is the process $(Y_n)_{n\geq 0}$ and $Y_n \geq 0$.

Answers:

(a)

$$\begin{aligned} E\big(Y_{n+1} \mid \mathcal{F}_n\big) &= E\big(X_{n+1} \mid \mathcal{F}_n\big) + E\big(Y_n \mid \mathcal{F}_n\big) \\ &= E(X_{n+1}) + Y_n \\ &= 2p + (-1)(1-p) + Y_n \\ &= 3p - 1 + Y_n \ . \end{aligned} \tag{1}$$

The second identity is due to the independence of X_{n+1} and $\mathcal{F}_n$ plus the fact that Y_n is $\mathcal{F}_n$-measurable ("taking out what is known"). Therefore:

-i- $E\big(Y_{n+1} \mid \mathcal{F}_n\big) = Y_n$ *iff* $p = 1/3$.

-ii- $E\big(Y_{n+1} \mid \mathcal{F}_n\big) \geq Y_n$ *iff* $p \geq 1/3$.

-iii- $E\big(Y_{n+1} \mid \mathcal{F}_n\big) \leq Y_n$ *iff* $p \leq 1/3$.

(b) Let F be measurable function, then

$$\begin{aligned} E\big(F(Y_{n+1}) \mid \mathcal{F}_n\big) &= E\big(F(X_{n+1} + Y_n) \mid \mathcal{F}_n\big) \\ &= pF(2 + Y_n) + (1-p)F(-1 + Y_n) \\ &= g(Y_n) \end{aligned}$$

with $g(s) = pF(2+s) + (1-p)F(-1+s)$.

(c) By (conditioned) Jensen, as $f(x) = x^4$ is convex,

$$\begin{aligned} E\big(Y_{n+1}^4 \mid \mathcal{F}_n\big) &\geq E\big(Y_{n+1} \mid \mathcal{F}_n\big)^4 \\ &\geq Y_{n+1}^4 \end{aligned}$$

the last inequality following from the sub-martingale character of $(Y_n)_{n\geq 0}$.

Exercise 3. [Filtrations and (non-)stopping times] Consider the ternary version of the two-period binary scenario discussed in class. This corresponds to the sample space $\Omega = \big\{(\omega_1, \omega_2) : \omega_i \in \{1,2,3\}\big\}$ with the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$, where $\mathcal{F}_0$ is formed only by the empty set and Ω, $\mathcal{F}_1$ formed by all events depending only on the first number, and $\mathcal{F}_2$ all events in Ω. Let $\tau_i : \Omega \to \{0, 1, 2, \infty\}$ be the functions defined by the following table:

ω_1	ω_2	τ_1	τ_2
1	1	∞	∞
1	2	2	2
1	3	∞	∞
2	1	1	1
2	2	1	2
2	3	1	1
3	1	∞	∞
3	2	2	2
3	3	∞	∞

(a) (0.7 pt.) Show that τ_1 is a stopping time.

(b) (0.7 pt.) Show that τ_2 is *not* a stopping time.

Answers:

(a) We must prove that

$$\{\tau_1 = n\} \in \mathcal{F}_n \quad , n = 0, 1 \tag{2}$$

(every event is in $\mathcal{F}_2$). Let us recall that

$$\begin{aligned} \mathcal{F}_0 &= \Big\{\emptyset, \Omega\Big\} \\ \mathcal{F}_1 &= Bigl\{\emptyset, \big\{(1,1),(1,2),(1,3)\big\}, \big\{(2,1),(2,2),(2,3)\big\}, \Omega\Big\} . \end{aligned} \tag{3}$$

Indeed we have

$$\begin{aligned} \{\tau_1 = 0\} &= \emptyset \in \mathcal{F}_0 \\ \{\tau_1 = 1\} &= \big\{(2,1),(2,2),(2,3)\big\} \in \mathcal{F}_1 . \end{aligned}$$

(b) We have

$$\{\tau_1 = 1\} = \big\{(2,1),(2,3)\big\} \notin \mathcal{F}_1$$

hence (2) is violated for $n = 1$.

In fact τ_1 =first time a "2" appears, while τ_1 =last time a "2" appears. The former can be decided on the basis of the information available (namely, I got a "2" or not), while the latter requires looking into the future and hence it can not be a stopping time.

Exercise 4. [European vs American options] A stock whose present value is $S_0 = 4$ evolves following a binomial model with $u = 1.5$ and $d = 1/2$; both possibilities having equal probability. The interest rate for each period is 5%.

(a) (0.4 pt.) Determine the risk-neutral probability for three periods.

(b) A European call option is established for 3 periods with strike value $K = S_0$. The final payoff of such an option depends on the maximum price of the stock in the last two periods:

$$V_3 = \big|\max\{S_2, S_3\} - S_0\big|_+ .$$

Determine

-i- (0.7 pt.) The fair price of the option.

-ii- (0.7 pt.) The hedging strategy for the seller.

(c) As an alternative, the financial institution offers the American version of the option, namely an option with intrinsic payoff

$$\begin{aligned} G_0 &= 0 \\ G_n &= \max\{S_{n-1}, S_n\} - S_0 \quad , \; n = 1, 2, 3 . \end{aligned}$$

Determine

-i- (0.7 pt.) The fair price of the option.

-ii- (0.7 pt.) The hedging strategy for the seller.

-iii- (0.7 pt.) The optimal exercise times for the buyer.

(d) (0.5 pt.) One of the criteria to decide which option is more convenient is to compare the expected *net market* payoff for each option. That is, the market average of the payoff (at the optimal times) minus the initial payment, with all values translated to the end of the 3rd period. Which of the options would you recommend on the basis of this criterion?

(e) (0.5 pt.) A theorem was discussed in class proving that the optimal exercise time for some American call options is at the last period or never, so they end up being no different than the European version. Explain why this theorem does not apply to the option in part (c).

Answers: *The market scenario is binomial, with*

$$\begin{array}{llll} & & & S_3(HHH) = 13.5 \\ & & S_2(HH) = 9 & \\ & S_1(H) = 6 & & S_3(HHT) = S_3(HTH) = S_3(THH) = 4.5 \\ S_0 = 4 & & S_2(HT) = S_2(TH) = 3 & \\ & S_1(T) = 2 & & S_3(HTT) = S_3(THT) = S_3(TTH) = 1.5 \\ & & S_2(TT) = 1 & \\ & & & S_3(TTT) = 0.5 \end{array}$$

(a) The risk-neutral probability is defined by

$$\widetilde{p}_n = \frac{1+r-d}{u-d} = \frac{1.05-0.5}{1.5-0.5} = 0.55$$

independent of n. More explicitly, it is defined by

$$\begin{aligned} \widetilde{P}\big(\{(H,H,H)\}\big) &= 0.55^3 \\ \widetilde{P}\big(\{(H,H,T)\}\big) &= \widetilde{P}\big(\{(H,T,H)\}\big)\widetilde{P}\big(\{(T,H,H)\}\big) = 0.55^2 \cdot 0.45 \\ \widetilde{P}\big(\{(H,T,T)\}\big) &= \widetilde{P}\big(\{(T,T,H)\}\big)\widetilde{P}\big(\{(T,H,T)\}\big) = 0.45^2 \cdot 0.55 \\ \widetilde{P}\big(\{(H,H,H)\}\big) &= 0.45^3 \end{aligned}$$

(b) -i- *The values of V_3 are:*

History	V_3
HHH	*9.5*
HHT	*5*
HTH	*0.5*
HTT	*0*
THH	*0.5*
THT	*0*
TTH	*0*
TTT	*0*

. (4)

Hence,

$$\begin{aligned} V_0 &= \widetilde{E}(\overline{V}_3) \\ &= \frac{1}{1.05^3}\big[0.55^3 \cdot 9.5 + 0.55^2 \cdot 0.45(5+0.5+0.5)\big] \\ &\approx 2.07\,. \end{aligned}$$

-ii- *Using, successively, the formulas*

$$V_2(\omega_1,\omega_2) = \frac{1}{1.05}\big[0.55\,V_3(\omega_1,\omega_2,H) + 0.45\,V_3(\omega_1,\omega_2,T)\big]$$

and

$$V_1(\omega_1) = \frac{1}{1.05}\big[0.55\,V_2(\omega_1,H) + 0.45\,V_2(\omega_1,T)\big]$$

we get (keeping only two decimal places)

History	V_2
HH	*7.12*
HT	*0.26*
TH	*0.26*
TT	*0*

,

History	V_1
H	*3.84*
T	*0.14*

. (5)

The formulas

$$\Delta_n(\omega_1,\ldots,\omega_n) \;=\; \frac{V_{n+1}(\omega_1,\ldots,\omega_n,H)-V_{n+1}(\omega_1,\ldots,\omega_n,T)}{S_{n+1}(\omega_1,\ldots,\omega_n,H)-S_{n+1}(\omega_1,\ldots,\omega_n,T)}\,.$$

yield, from (4) and (5),

History	Δ_2
HH	*0.5*
HT	*0.17*
TH	*0.17*
TT	*0*

,

History	Δ_1
H	*1.14*
T	*0.13*

, $\Delta_0 \;=\; 0.93\,.$ (6)

(c) -i- *The values of* V_3 *and* G_3 *are:*

History	V_3	G_3	$V_3 = G_3$
HHH	*9.5*	*9.5*	*
HHT	*5*	*5*	*
HTH	*0.5*	*0.5*	*
HTT	*0*	*<0*	
THH	*0.5*	*0.5*	*
THT	*0*	*<0*	
TTH	*0*	*<0*	
TTT	*0*	*<0*	

. (7)

-ii- *We use, successively, the formulas*

$$\begin{aligned}\widetilde{E}\big(V_{n+1}/R \mid \mathcal{F}_n\big)(\omega_1,\ldots,\omega_n) &= \frac{1}{1.05}\big[0.55\,V_{n+1}(\omega_1,\ldots,\omega_n,H)+0.45\,V_{n+1}(\omega_1,\ldots,\omega_n,T)\big]\\ G_n &= \max\{S_{n-1},S_n\}-S_0\end{aligned}$$

and

$$V_n \;=\; \max\Big\{\widetilde{E}\big(V_{n+1}/R \mid \mathcal{F}_n\big), G_n\Big\}$$

for $n = 2,1,0$. *We get (keeping only two decimal places)*

History	$\widetilde{E}(V_3/R \mid \mathcal{F}_2)$	G_2	V_2	$V_2 = G_2$
HH	*7.12*	*5*	*7.12*	
HT	*0.26*	*2*	*2*	*
TH	*0.26*	*0*	*0.26*	
TT	*0*	*0*	*0*	*

, (8)

History	$\widetilde{E}(V_2/R \mid \mathcal{F}_1)$	G_1	V_1	$V_1 = G_1$
H	*4.59*	*2*	*4.59*	
T	*0.14*	*0*	*0.14*	

(9)

and, as $G_0 = 0$,

$$V_0 \;=\; \widetilde{E}(V_1/R) \;=\; \frac{1}{1.05}\big[0.55\cdot 4.59+0.45\cdot 0.14\big] \;\approx\; 2.46$$

-iii- *The formulas*

$$\Delta_n(\omega_1,\ldots,\omega_n) \;=\; \frac{V_{n+1}(\omega_1,\ldots,\omega_n,H)-V_{n+1}(\omega_1,\ldots,\omega_n,T)}{S_{n+1}(\omega_1,\ldots,\omega_n,H)-S_{n+1}(\omega_1,\ldots,\omega_n,T)}\;.$$

yield, from (7), (8) and (9),

History	Δ_2
HH	*0.5*
HT	*0.17*
TH	*0.17*
TT	*0*

,

History	Δ_1
H	*0.85*
T	*0.13*

, $\Delta_0 \;=\; 1.11\;.$ (10)

-iv- *The time is determined by the condition:*

$$\tau \;=\; \begin{cases} \infty & \text{if } V_n \neq G_n\ \forall n \\ \min\{n : V_n = G_n\} & \text{otherwise}\,. \end{cases}$$

The case $V_n = G_n = 0$ is, in fact, undetermined because it is equally unfavourable to exercise or not. We adopt the usual choice of not exercising at these times. Looking into the "∗" in (7)–(8) we conclude that

τ	V_3
HHH	*3*
HHT	*3*
HTH	*2*
HTT	*2*
THH	*3*
THT	∞
TTH	∞
TTT	∞

.

(d) *The European (market) average final payoff is*

$$\frac{1}{8}(9.5+5+0.5+0.5) - 2.07\cdot 1.05^3 \;=\; -0.46\;.$$

The American (market) average final payoff is

$$\frac{1}{8}(9.5+5+0.5) + \frac{1}{4}2\times 1.05 - 2.46\cdot 1.05^3 \;=\; -0.45\;.$$

The choices are basically equivalent from this point of view.

(e) *The theorem applies for cases in which the final payoff is a function of S_N only.*

Bonus problem

Bonus . [Black-Scholes-Merton market] The BSM market with volatility rate σ is a log-normal market with distribution

$$S(t) \;=\; S(0)\,\mathrm{e}^{[\mu-(\sigma^2/2)]\,t+\sigma\sqrt{t}\,Y}$$

where Y is a standard normal random variable (normal variable with mean 0 and variance 1).

(a) (0.7 pt.) Prove that

$$E\big(S(t)\big) \;=\; S(0)\,\mathrm{e}^{\mu t}\;.$$

(b) (0.5 pt.) Show that μ is the *expected* instantaneous rate of return, that is, show

$$\mu \;=\; \lim_{t\to 0} \frac{1}{t} E\Big(\frac{S(t)-S(0)}{S(0)}\Big) \,.$$

Answers:

(a)

$$\begin{aligned} E\big(S(t)\big) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{[\mu-(\sigma^2/2)]\,t+\sigma\sqrt{t}\,y}\, \mathrm{e}^{-y^2/2}\, dy \\ &= \mathrm{e}^{\mu t}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(y-\sigma)^2/2}\, dy \\ &= \mathrm{e}^{\mu t}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\, dx \\ &= \mathrm{e}^{\mu t} \end{aligned}$$

(b) Expanding the exponential,

$$E\big(S(t)\big) \;=\; S(0)\,\big(1+\mu t + o(t)\big) \,.$$

Therefore,

$$\frac{1}{t} E\Big(\frac{S(t)-S(0)}{S(0)}\Big) \;=\; \mu + \frac{o(t)}{t} \xrightarrow[t\to 0]{} \mu \,.$$