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Deeltentamen 2 Inleiding Financiele Wiskunde, 2011-12

exercise:	1	2	3	4
points:	25	25	25	25

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.1$, $d = 0.8$, and $r = 0.05$. Consider an American Put option with expiration $N = 2$ and strike price $K = 90$.

   (a) Determine the price V_n at time $n = 0, 1$ of the American put option.

   (b) Determine the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

   (c) Suppose $\omega_1\omega_2 = TT$. Find the values of the replicating portfolio process $\Delta_0, \Delta_1(T)$. Show that if the buyer exercises at time 1, then $X_1(T) = V_1(T)$, and if the buyer exercises at time 2, then $X_2(TT) = V_2(TT)$.

Solution (a): Note that the risk neutral probbaility is $\widetilde{p} = 5/6$ and $\widetilde{q} = 1/1$. The price process is given by

$S_0 = 100, S_1(H) = 110, S_1(T) = 80, S_2(HH) = 121, S_2(HT) = S_2(TH) = 88, S@(TT) = 64.$

The intrinsic value process is given by

$$G_0 = -10, G_1(H) = -20, G_1(T) = 10,$$

$$G_2(HH) = -31, G_2(HT) = 2, G_2(TH) = 2, G_2(TT) = 26.$$

The payoff at time 2 is given by

$$V_2(HH) = 0,\ V_2(HT) = 2,\ V_2(TH) = 2,\ V_2(TT) = 26.$$

Applying the American algorithm, we get

$$V_1(H) = \max\left(-20, \frac{1}{1.05}[\frac{5}{6} \times 0 + \frac{1}{6} \times 2]\right) = 0.31746.$$

$$V_1(T) = \max\left(10, \frac{1}{1.05}[\frac{5}{6} \times 2 + \frac{1}{6} \times 26]\right) = \max(10, 5.71429) = 10.$$

$$V_0 = \max\left(-10, \frac{1}{1.05}[\frac{5}{6} \times 0.31746 + \frac{1}{6} \times 10]\right) = \max(-10, 1.83925) = 1.83925.$$

Solution (b): The optimal exercise time is given by

$$\tau^*(HH) = \infty,\ \tau^*(HT) = 2,\ \tau^*(TH) = \tau^*(TT) = 1.$$

Solution (c): We first calculate

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = -0.32275,$$

and

$$\Delta_1(T) = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} = -1.$$

The replicating portfolio is described as follows. At time 0 sell the option for $V_0 = 1.83925$, and short sell 0.32275 of a stock for 32.275. Deposit the proceeds in the money market. So

$$X_0 = 1.83925 = \Delta_0 S_0 + 34.11425.$$

At time 1, $\omega_1 = T$, so your wealth equals

$$X_1(T) = \Delta_0 S_1(T) + (1.05)(34.11425) = 10 = V_1(T).$$

If the option is exercised at time 1, then the payoff is 10, and this is equal to your wealth. If the option is not exercised, then you adjust your portfolio (without changing your wealth), so

$$X_1(T) = 10 = \Delta_1(T) S_1(T) + 90,$$

and you can consume $C_1(T) = V_1(T) - \widetilde{E}_1((1.05)^{-1}V_2)(T) = 10 - 5.71429 = 4.28571$. At time 2 $\omega_2 = T$, so

$$X_2(TT) = \Delta_1(T) S_2(TT) + (1.05)(90 - 4.28571) = 26 = V_2(TT).$$

2. Consider the binomial model with up factor $u = 2$, down factor $d = 1/2$ and interest rate $r = 1/4$. Consider a perpetual American put option with $S_0 = 2^j$, and $K = S_0 2^{-m}$. Suppose that the buyer of the option exercises the first time the price is less than or equal to $K/2$.

   (a) Show that the price at time zero of this option is given by

$$V_0 = \begin{cases} K - S_0, & \text{if } S_0 \le K/2, \\ \dfrac{K^2}{4S_0}, & \text{if } S_0 \ge K. \end{cases}$$

   (b) Consider the process $v(S_0), v(S_1), \cdots$ defined by

$$v(S_n = \begin{cases} K - S_n, & \text{if } S_n \le K/2, \\ \dfrac{K^2}{4S_n}, & \text{if } S_n \ge K. \end{cases}$$

   Show that $v(S_n) \ge (K - S_n)^+$ for all $n \ge 0$, and that the discounted process $\{\left(\frac{4}{5}\right)^n v(S_n) : n = 0, 1, \cdots\}$ is a supermartingale.

Solution (a): If $S_0 \le K/2$, then the buyer exercises immediately. His payoff is $K - S_0$. So the price in this case must be $K - S_0$ as required. If $S_0 \ge K$, then the buyer uses the exercise policy $\tau_{-(m+1)}$. Note that $S_{\tau_{-(m+1)}} = K/2$, and by Theorem 5.2.3 we have $\left(\frac{4}{5}\right)^{\tau_{-(m+1)}} = \left(\frac{1}{2}\right)^{m+1}$.

So the price of the option in this case is

$$\begin{aligned} V_0 = V^{\tau_{-(m+1)}} &= \widetilde{E}\left(\left(\frac{4}{5}\right)^{\tau_{-(m+1)}} (K - S_{\tau_{-(m+1)}})\right) \\ &= \frac{K}{2}\left(\frac{1}{2}\right)^{m+1} = \frac{K S_0 2^{-m}}{4S_0} = \frac{K^2}{4S_0}. \end{aligned}$$

Solution (b): We first show that $v(S_n) \ge (K - S_n)^+$. If $S_n \le K/2$, then $v(S_n) = (K - S_n) = (K - S_n)^+$. If $S_n \ge K$, then $v(S_n) = \frac{K^2}{4S_n} > 0 = (K - S_n)^+$. We now show that the discounted process is a supermartingale. If $S_n < K/2$, then $S_{n+1} \le K/2$, thus

$$\begin{aligned} \widetilde{E}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) &= \left(\frac{4}{5}\right)^{n+1} [\frac{1}{2}(K - 2S_n) + \frac{1}{2}(K - S_n/2)] \\ &= \left(\frac{4}{5}\right)^{n} (\frac{4}{5}K - S_n) \\ &< \left(\frac{4}{5}\right)^{n} (K - S_n) = \left(\frac{4}{5}\right)^{n} v(S_n). \end{aligned}$$

If $S_n = K/2$, then $S_{n+1} \in \{K/4, K\}$. Thus,

$$\begin{aligned} \widetilde{E}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) &= \left(\frac{4}{5}\right)^{n+1} [\frac{1}{2}\frac{K}{4} + \frac{1}{2}\frac{3K}{4}] \\ &= \left(\frac{4}{5}\right)^{n} \frac{2}{5}K \\ &< \left(\frac{4}{5}\right)^{n} \frac{1}{2}K = \left(\frac{4}{5}\right)^{n} v(S_n). \end{aligned}$$

If $S_n \ge K$, then

$$\begin{aligned} \widetilde{E}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) &= \left(\frac{4}{5}\right)^{n+1} [\frac{1}{2}\frac{K^2}{8S_n} + \frac{1}{2}\frac{K^2}{2S_n}] \\ &= \left(\frac{4}{5}\right)^{n} \frac{K^2}{4S_n} = \left(\frac{4}{5}\right)^{n} v(S_n). \end{aligned}$$

In all cases we have $\widetilde{E}_n\left(\left(\frac{4}{5}\right)^{n+1} v(S_{n+1})\right) \le \left(\frac{4}{5}\right)^{n} v(S_n)$ as required.

3. Consider a random walk $M_0, M_1, \cdots$ with probability p for an up step and $q = 1 - p$ for a down step, $0 < p < 1$. For $a \in \mathbb{R}$, define $S_n^a = 10^{-n + aM_n}$, $n = 0, 1, 2, \cdots$.

(a) For which values of a is the the process $S_0^a, S_1^a, \cdots$ a martingale?

(b) Suppose now that $p = 1/2$, so $M_0, M_1, \cdots$, is the symmetric random walk. Let $\tau_m = \inf\{n \geq 0 : M_n = m\}$. Determine the value of $E(S_{\tau_m}^a)$.

Solution (a): First note that the process (S_n^a) is adjusted, and

$$S_{n+1}^a = 10^{-n-1+aM_n+aX_{n+1}} = S_n^a\, 10^{aX_{n+1}-1}.$$

Since X_{n+1} is independent of the first n tosses we have

$$E_n(10^{aX_{n+1}-1}) = E(10^{aX_{n+1}-1}) = 10^{-1}(10^a\, p + 10^{-a}\, q).$$

Thus,

$$E_n(S_{n+1}^a) = S_n^a 10^{-1}(10^a\, p + 10^{-a}\, q).$$

For the process to be a martingale, we need to find the values of a such that

$$10^{-1}(10^a\, p + 10^{-a}\, q) = 1$$

or equivalently,

$$p10^{2a} - 10\, 10^a + q = 0.$$

Solving, we get

$$10^a = \frac{5 \pm \sqrt{25 - pq}}{p}$$

implying

$$a = \log_{10}\left(\frac{5 \pm \sqrt{25 - pq}}{p}\right).$$

Solution (c): Observe that $S_{\tau_m}^a = 10^{-\tau_m + aM_{\tau_m}} = 10^{-\tau_m + ma}$. By Theorem 5.2.3 we have

$$E(S_{\tau_m}^a) = 10^{ma} E((\frac{1}{10})^{\tau_n}) = 10^{ma}(10 - 3\sqrt{11})^m.$$

4. Consider a 3-period (non constant interest rate) binomial model with interest rate process R_0, R_1, R_2 defined by

$$R_0 = 0,\ R_1(\omega_1) = .05 + .01H_1(\omega_1),\ R_2(\omega_1, \omega_2) = .05 + .01H_2(\omega_1, \omega_2)$$

where $H_i(\omega_1, \cdots, \omega_i)$ equals the number of heads appearing in the first i coin tosses $\omega_1, \cdots, \omega_i$. Suppose that the risk neutral measure is given by $\widetilde{P}(HHH) = \widetilde{P}(HHT) = 1/8$, $\widetilde{P}(HTH) = \widetilde{P}(THH) = \widetilde{P}(THT) = 1/12$, $\widetilde{P}(HTT) = 1/6$, $\widetilde{P}(TTH) = 1/9$, $\widetilde{P}(TTT) = 2/9$.

(a) Calculate $B_{1,2}$ and $B_{1,3}$, the time one price of a zero coupon maturing at time two and three respectively.

(b) Consider a 3-period interest rate swap. Find the 3-period swap rate SR_3, i.e. the value of K that makes the time zero no arbitrage price of the swap equal to zero.

(c) Consider a 3-period floor that makes payments $F_n = (.055 - R_{n-1})^+$ at time $n = 1, 2, 3$. Find Floor$_3$, the price of this floor.

Solution (a): We first calcultate the values of R_0, R_1, R_2 and D_1, D_2, D_3 in the following tables:

$\omega_1\omega_2$	R_0	R_1	R_2
HH	0	0.06	0.07
HT	0	0.06	0.06
TH	0	0.05	0.06
TT	0	0.05	0.05

$\omega_1\omega_2$	$\frac{1}{1+R_0}$	$\frac{1}{1+R_1}$	$\frac{1}{1+R_2}$	D_1	D_2	D_3	$\widetilde{P}$
HH	1	$\frac{1}{1.06}$	$\frac{1}{1.07}$	1	$\frac{1}{1.06}$	$\frac{1}{1.1342}$	$\frac{1}{4}$
HT	1	$\frac{1}{1.06}$	$\frac{1}{1.06}$	1	$\frac{1}{1.06}$	$\frac{1}{1.1236}$	$\frac{1}{4}$
TH	1	$\frac{1}{1.05}$	$\frac{1}{1.06}$	1	$\frac{1}{1.05}$	$\frac{1}{1.113}$	$\frac{1}{6}$
TT	1	$\frac{1}{1.05}$	$\frac{1}{1.05}$	1	$\frac{1}{1.05}$	$\frac{1}{1.1025}$	$\frac{1}{3}$

Since $D_1 = 1$ and D_2 is known at time 1, then $B_{1,2} = \widetilde{E}_1(D_2) = D_2$. This gives $B_{1,2}(H) = 1/1.06$ and $B_{1,2}(T) = 1/1.05$.

Now, D_3 depends on the first two coin tosses only, and since $D_1 = 1$ we have

$$\begin{aligned} B_{1,3}(H) = \widetilde{E}_1(D_3)(H) &= D_3(HH)\widetilde{P}(\omega_2 = H|\omega_1 = H) + D_3(HT)\widetilde{P}(\omega_2 = T|\omega_1 = H) \\ &= \frac{1}{1.1342}\frac{1}{2} + \frac{1}{1.1236}\frac{1}{2} = 0.8858, \end{aligned}$$

and

$$\begin{aligned} B_{1,3}(T) = \widetilde{E}_1(D_3)(T) &= D_3(TH)\widetilde{P}(\omega_2 = H|\omega_1 = T) + D_3(TT)\widetilde{P}(\omega_2 = T|\omega_1 = T) \\ &= \frac{1}{1.113}\frac{1}{3} + \frac{1}{1.1025}\frac{2}{3} = 0.9499. \end{aligned}$$

Solution (b): From Theorem 6.3.7, we know that

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}}.$$

Now,

$$B_{0,1} = \widetilde{E}(D_1) = 1,$$

D_2 depends on the ω_1 only, so

$$\begin{aligned} B_{0,2} = \widetilde{E}(D_2) &= \frac{1}{1.06}\widetilde{P}(\omega_1 = H) + \frac{1}{1.05}\widetilde{P}(\omega_1 = T) \\ &= \frac{1}{1.06}\frac{1}{2} + \frac{1}{1.05}\frac{1}{2} = 0.94789, \end{aligned}$$

Now, D_3 depends only on ω_1, ω_2, hence

$$\begin{aligned}
B_{0,3} = \widetilde{E}(D_3) &= \frac{1}{1.1342}\widetilde{P}(\omega_1 = H, \omega_2 = H) + \frac{1}{1.1236}\widetilde{P}(\omega_1 = H, \omega_2 = T) \\
&+ \frac{1}{1.113}\widetilde{P}(\omega_1 = T, \omega_2 = H) + \frac{1}{1.1025}\widetilde{P}(\omega_1 = H, \omega_2 = H) \\
&= \frac{1}{1.1342}\frac{1}{4} + \frac{1}{1.1236}\frac{1}{4} + \frac{1}{1.113}\frac{1}{6} + \frac{1}{1.1025}\frac{1}{3} \\
&= 0.895.
\end{aligned}$$

Thus,

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}} = \frac{1 - 0.91787}{2.86576} = 0.0287.$$

Solution (c): From Definition 6.3.8 we have

$$Floor_3 = \sum_{n=1}^{3} \widetilde{E}(D_n(0.055 - R_{n-1})^+.)$$

We display the values of $(0.055 - R_{n-1})^+)$ in a table

$\omega_1\omega_2$	$(0.055 - R_0)^+$	$(0.055 - R_1)^+$	$(0.055 - R_2)^+$
HH	0.055	0	0
HT	0.055	0	0
TH	0.055	0.005	0
TT	0.055	0.005	0.005

Thus,

$$\widetilde{E}(D_1(0.055 - R_0)^+) = 0.055,$$

$$\widetilde{E}(D_2(0.055 - R_1)^+) = D_2(H)(0)P(H) + D_2(T)(0.005)P(T) = \frac{1}{1.05}(0.005)\frac{1}{2} = 0.00238,$$

and

$$\widetilde{E}(D_3(0.055 - R_2)^+) = D_3(TT)(0.055)P(TT) = \frac{1}{1.1025}(0.005)\frac{1}{3} = 0.00151$$

Therefore,

$$Floor_3 = 0.055 + 0.00238 + 0.00151 = 0.05889.$$