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Hertentamen Inleiding Financiele Wiskunde, 2011-12

* Punten per opgave:

opgave:	1	2	3	4
punten:	30	20	20	30

1. Consider a 2-period binomial model with $S_0 = 20$, $u = 1.3$, $d = 0.9$, and $r = 0.1$. Suppose the real probability measure P satisfies $P(H) = p = \dfrac{1}{3} = 1 - P(T)$.

   (a) Consider an Asian European option with payoff $V_2 = ((S_1 + S_2)/2 - 20)^+$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = HT$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ so that the corresponding wealth process satisfies $X_0 = V_0$ (your answer in part (a)) and $X_2(HT) = V_2(HT)$.

   (c) Consider the utility function $U(x) = 4x^{1/4}$ $(x > 0)$. Show that the random variable $X = X_2$ (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = X_0$ is given by

   $$X = X_2 = \frac{(1.1)^2 X_0}{Z^{4/3} E(Z^{-1/3})}.$$

   (d) Consider part (c) and assume $X_0 = 20$. Determine the value of the optimal portfolio process $\{\Delta_0, \Delta_1\}$ and the value of the corresponding wealth process $\{X_0, X_1, X_2\}$.

   (e) Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 20 - \dfrac{S_0 + \cdots + S_n}{n+1}$, $n = 0, 1, 2$. Determine the price V_n at time $n = 0, 1$ of the American option. Find the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

2. Consider a 3-period (non constant interest rate) binomial model with interest rate process R_0, R_1, R_2 defined by

   $$R_0 = 0,\ R_1(\omega_1) = .05 + .01H_1(\omega_1),\ R_2(\omega_1, \omega_2) = .05 + .01H_2(\omega_1, \omega_2)$$

   where $H_i(\omega_1, \cdots, \omega_i)$ equals the number of heads appearing in the first i coin tosses $\omega_1, \cdots, \omega_i$. Suppose that the risk neutral measure is given by $\widetilde{P}(HHH) = \widetilde{P}(HHT) = 1/8$, $\widetilde{P}(HTH) = \widetilde{P}(THH) = \widetilde{P}(THT) = 1/12$, $\widetilde{P}(HTT) = 1/6$, $\widetilde{P}(TTH) = 1/9$, $\widetilde{P}(TTT) = 2/9$.

(a) Calculate $B_{1,2}$ and $B_{1,3}$, the time one price of a zero coupon maturing at time two and three respectively.

(b) Consider a 3-period interest rate swap. Find the 3-period swap rate SR_3, i.e. the value of K that makes the time zero no arbitrage price of the swap equal to zero.

(c) Consider a 3-period floor that makes payments $F_n = (.055 - R_{n-1})^+$ at time $n = 1, 2, 3$. Find Floor_3, the price of this floor.

3. Consider the binomial model with $u = 2^1$, $d = 2^{-1}$, and $r = 1/4$, and consider a perpetual American put option with $S_0 = 10$ and $K = 12$. Suppose that Alice and Bob each buy such an option

(a) Suppose that Alice uses the strategy of exercising the first time the price reaches 5 euros. What should then the price be at time 0?

(b) Suppose that Bob uses the strategy of exercising the first time the price reaches 2.5 euros. What should then the price be at time 0?

(c) What is the probability that the price reaches 20 euros for the first time at time $n = 5$?

4. Consider a random walk $M_0, M_1, \cdots$ with probability p for an up step and $q = 1-p$ for a down step, $0 < p < 1$. For $a \in \mathbb{R}$ and $b > 1$, define $S_n^a = b^{-n}2^{aM_n}$, $n = 0, 1, 2, \cdots$.

(a) For which values of a is the the process $S_0^a, S_1^a, \cdots$ a (i) martingale, (ii) supermartingale, (iii) submartingale?

(b) Show that the process $S_0^a, S_1^a, \cdots$ is a Markov Process.

(c) Suppose now that $p = 1/2$, so $M_0, M_1, \cdots$, is the symmetric random walk. Let $\tau_m = \inf\{n \geq 0 : M_n = m\}$. Determine the value of $E(S_{\tau_m}^a)$.