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OefenDeeltentamen 1 Inleiding Financiele Wiskunde, 2011-12

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.5$, $d = 0.5$, and $r = 0.25$. Suppose the real probability measure P satisfies $P(H) = p = \dfrac{2}{3} = 1 - P(T)$.

   (a) Consider an option with payoff $V_2 = \left(\dfrac{S_1 + S_2}{2} - 105\right)^+$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = HT$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ so that so that the corresponding wealth process satisfies $X_0 = V_0$ (your answer in part (a)) and $X_2(HT) = V_2(HT)$.

   (c) Determine explicitly the Radon-Nikodym process Z_0, Z_1, Z_2, where

   $$Z_2(\omega_1\omega_2) = Z(\omega_1\omega_2) = \frac{\widetilde{P}(\omega_1\omega_2)}{P(\omega_1\omega_2)}$$

   with $\widetilde{P}$ the risk neutral probability measure, and $Z_i = E_i(Z)$, $i = 0, 1,$.

   (d) Consider the utility function $U(x) = \ln x$. Find a random variable X (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = 30$. Find the corresponding optimal portfolio process $\{\Delta_0, \Delta_1\}$.

2. Consider the N-period binomial model, and assume that $P(H) = P(T) = 1/2$ (we use the same notation as the book). Set $X_0 = 0$, and define for $n = 1, 2, \cdots, N$

   $$X_i(\omega_1 \dots \omega_N) = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

   and set $S_n = \sum_{i=0}^{N} X_i$, $n = 0, 1, \cdots, N$.

   (a) Let $Y_n = S_n^2$, $n = 0, 1, \cdots, N$. Show that $E_n(Y_{n+1}) = 1 + Y_n$, $n = 0, 1, \cdots, N-1$. Conclude that the process $Y_0, Y_1, \cdots, Y_N$ is a submartingale with respect to P.

   (b) Let $Z_n = Y_n - n$, $n = 0, 1, \cdots, N$. Show that the process $Z_0, Z_1, \cdots, Z_N$ is a martingale with respect to P

   (c) Let $a > 0$, and define $U_n = a^{S_n}\left(\dfrac{a^2+1}{2a}\right)^{-n}$. Show that the process

   $$U_0, U_1, \cdots, U_N$$

   is a martingale w.r.t. P.

3. Consider the N-period binomial model, and assume that $P(H) = P(T) = 1/2$ (we use the same notation as the book).

   (a) Assume $X_0, X_1, \ldots, X_N$ is a Markov process w.r.t. the risk neutral measure $\widetilde{P}$. Consider an option with payoff $V_N = X_N^2$. Show that for each $n = 0, 1, \ldots, N-1$, there exists a function g_n such that the price at time n is given by $V_n = g_n(X_n)$.

   (b) Let $X_0, X_1, \ldots, X_N$ be an adapted process on (Ω, P). Consider the random variables $U_1, \ldots, U_N$ on (Ω, P) defined by

$$U_i(\omega_1 \ldots \omega_N) = \begin{cases} 1/2, & \text{if } \omega_i = H, \\ -1/2, & \text{if } \omega_i = T. \end{cases}$$

   Let $Z_0 = 0$, and $Z_n = \sum_{j=0}^{n-1} X_j U_{j+1}$, $n = 1, 2, \ldots, N$. Prove that the process $Z_0, Z_1, \ldots, Z_N$ is a martingale w.r.t. P.

   (c) Consider the process $U_1, \ldots, U_N$ of part (b), and define

$$S_n = \sum_{i=1}^{n} U_i, \text{ and } M_n = \min_{1 \le i \le n} S_n,$$

   voor $n = 1, 2, \cdots, N$. Show that the process $(M_1, S_1), \cdots (M_N, S_N)$ is Markov w.r.t. P.