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OefenDeeltentamen 1 Inleiding Financiele Wiskunde, 2011-12

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.5$, $d = 0.5$, and $r = 0.25$. Suppose the real probability measure P satisfies $P(H) = p = \dfrac{2}{3} = 1 - P(T)$.

   (a) Consider an option with payoff $V_2 = \left(\dfrac{S_1 + S_2}{2} - 105\right)^+$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = HT$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ so that so that the corresponding wealth process satisfies $X_0 = V_0$ (your answer in part (a)) and $X_2(HT) = V_2(HT)$.

   (c) Determine explicitly the Radon-Nikodym process Z_0, Z_1, Z_2, where

   $$Z_2(\omega_1\omega_2) = Z(\omega_1\omega_2) = \frac{\widetilde{P}(\omega_1\omega_2)}{P(\omega_1\omega_2)}$$

   with $\widetilde{P}$ the risk neutral probability measure, and $Z_i = E_i(Z)$, $i = 0, 1,$.

   (d) Consider the utility function $U(x) = \ln x$. Find a random variable X (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = 30$. Find the corresponding optimal portfolio process $\{\Delta_0, \Delta_1\}$.

Solution (a): The risk nuetral measure is given by $\widetilde{p} = 2/3 = 1 - \widetilde{q}$, and the payoff at time 2 is:

$$V_2(HH) = 82.5,\ V_2(HT) = 7.5,\ V_2(TH) = V_2(TT) = 0.$$

Now,

$$V_1(H) = \frac{1}{1.25}[\frac{3}{4}(82.5) + \frac{1}{4}(7.5)] = 51,$$
$$V_1(T) = \frac{1}{1.25}[\frac{3}{4}(0) + \frac{1}{4}(0)] = 0,$$

and

$$V_1(H) = \frac{1}{1.25}[\frac{3}{4}(51) + \frac{1}{4}(0)] = 30.6.$$

Solution (b):

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = 0.51,$$

and

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = 0.5.$$

Solution (c): Notice that

$$Z_i(\omega_1, \cdots, \omega_i) = E_i(Z_2)(\omega_1, \cdots, \omega_i) = \frac{\widetilde{P}(\omega_1, \cdots, \omega_i)}{P(\omega_1, \cdots, \omega_i)}.$$

Thus,

$$Z_2(HH) = \frac{81}{64},\ Z_2(HT) = Z_2(TH) = \frac{27}{32},\ Z_2(TT) = \frac{9}{16},$$
$$Z_1(H) = \frac{9}{8},\ Z_1(T) = \frac{3}{4},\ Z_0 = 1.$$

Solution (d): The fastest way is to use Theorem 3.3.6. We find that $U'(x) = \frac{1}{x}$ and the inverse I of U' is also given by $I(x) = \frac{1}{x}$. Denoting the Radon Nikodym derivative by $Z = Z_2$, the solution X is given by

$$X = I\left(\frac{\lambda Z}{(1.25)^2}\right) = \frac{(1.25)^2}{\lambda Z}.$$

To find λ, we use the constraint

$$\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = E\left(\frac{ZX}{(1+r)^2}\right) = \frac{1}{\lambda} = 30.$$

Hence, $\lambda = 1/30$, and $X = \frac{46.875}{Z}$. That is,

$$X(HH) = 37.04,\ X(HT) = X(TH) = 55.56,\ X(TT) = 83.33.$$

The corresponding optimal portfolio can be found using Theorem 1.2.2. Writing $X_2 = X$,the wealth process is given by

$$X_1(H) = \widetilde{E}_1\left(\frac{X}{1.25}\right)(H) = \frac{1}{1.25}[\frac{3}{4}(37.04) + \frac{1}{4}(55.56)] = 33.336,$$
$$X_1(T) = \widetilde{E}_1\left(\frac{X}{1.25}\right)(T) = \frac{1}{1.25}[\frac{3}{4}(55.56) + \frac{1}{4}(83.33)] = 50.002,$$

and

$$X_0 = \frac{1}{1.25}[\frac{3}{4}(33.336) + \frac{1}{4}(50.002)] \approx 30,$$

as required (small discrepancy due to rounding off errors). The optimal portfolio process is given by

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = -0.16667,$$
$$\Delta_1(H) = \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_1(HT)} = -0.1235,$$

and

$$\Delta_1(T) = \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_1(TT)} = -0.5554.$$

2. Consider the N-period binomial model, and assume that $P(H) = P(T) = 1/2$ (we use the same notation as the book). Set $X_0 = 0$, and define for $n = 1, 2, \cdots, N$

$$X_i(\omega_1 \ldots \omega_N) = \begin{cases} 1, & \text{if } \omega_i = H, \\ -1, & \text{if } \omega_i = T, \end{cases}$$

and set $S_n = \sum_{i=0}^{N} X_i$, $n = 0, 1, \cdots, N$.

(a) Let $Y_n = S_n^2$, $n = 0, 1, \cdots, N$. Show that $E_n(Y_{n+1}) = 1 + Y_n$, $n = 0, 1, \cdots, N-1$. Conclude that the process $Y_0, Y_1, \cdots, Y_N$ is a submartingale with respect to P.

(b) Let $Z_n = Y_n - n$, $n = 0, 1, \cdots, N$. Show that the process $Z_0, Z_1, \cdots, Z_N$ is a martingale with respect to P

(c) Let $a > 0$, and define $U_n = a^{S_n} \left(\frac{a^2+1}{2a} \right)^{-n}$. Show that the process

$$U_0, U_1, \cdots, U_N$$

is a martingale w.r.t. P.

Solution (a): First note that $X_n^2 = 1$ for $n = 1, \cdots, n$ and $\widetilde{E}_n(X_{n+1}) = E(X_{n+1}) = 0$, this follows from the fact that X_{n+1} is independent from the first n tosses. Furthermore,

$$Y_{n+1} = S_{n+1}^2 = (S_n + X_{n+1})^2 = S_n^2 + 2S_n X_{n+1} + 1.$$

Thus,

$$E_n(Y_{n+1}) = S_n^2 + 2S_n E_n(X_{n+1}) + 1 = S_n^2 + 1 = Y_n + 1,$$

where we used the fact that at time n, S_n is known. Since $E_n(Y_{n+1}) = Y_n + 1 > Y_n$, we have that the process $Y_0, Y_1, \cdots, Y_N$ is a submartingale with respect to P.

Solution (b): Using again that $X_n^2 = 1$ for $n = 1, \cdots, n$, we have

$$Z_{n+1} = (S_n + X_{n+1})^2 - (n+1) = S_n^2 + 2S_n X_{n+1} - n.$$

Since S_n is known at time n and $\widetilde{E}_n(X_{n+1}) = E(X_{n+1}) = 0$, we have

$$E_n(Z_{n+1}) = S_n^2 + 2S_n E_n(X_{n+1}) - n = S_n^2 - n = Z_n.$$

Thus, $Z_0, Z_1, \cdots, Z_N$ is a martingale with respect to P.

Solution (c): First, we observe that

$$U_{n+1} = a^{S_n} \, a^{X_{n+1}} \left(\frac{a^2+1}{2a} \right)^{-(n+1)},$$

and

$$E_n(X_{n+1}) = E(X_{n+1}) = a/2 + a^{-1}/2 = \frac{a^2+1}{2a}.$$

Since S_n is known at time n, we have

$$E_n(U_{n+1}) = a^{S_n} \, E_n(a^{X_{n+1}}) \left(\frac{a^2+1}{2a} \right)^{-(n+1)} = a^{S_n} \left(\frac{a^2+1}{2a} \right)^{-n} = U_n.$$

Thus, $U_0, U_1, \cdots, U_N$ is a martingale w.r.t. P.

3. Consider the N-period binomial model, and assume that $P(H) = P(T) = 1/2$ (we use the same notation as the book).

   (a) Assume $X_0, X_1, \ldots, X_N$ is a Markov process w.r.t. the risk neutral measure $\widetilde{P}$. Consider an option with payoff $V_N = X_N^2$. Show that for each $n = 0, 1, \ldots, N-1$, there exists a function g_n such that the price at time n is given by $V_n = g_n(X_n)$.

   (b) Let $X_0, X_1, \ldots, X_N$ be an adapted process on (Ω, P). Consider the random variables $U_1, \ldots, U_N$ on (Ω, P) defined by

$$U_i(\omega_1 \ldots \omega_N) = \begin{cases} 1/2, & \text{if } \omega_i = H, \\ -1/2, & \text{if } \omega_i = T. \end{cases}$$

   Let $Z_0 = 0$, and $Z_n = \sum_{j=0}^{n-1} X_j U_{j+1}$, $n = 1, 2, \ldots, N$. Prove that the process $Z_0, Z_1, \ldots, Z_N$ is a martingale w.r.t. P.

   (c) Consider the process $U_1, \ldots, U_N$ of part (b), and define

$$S_n = \sum_{i=1}^{n} U_i, \text{ and } M_n = \min_{1 \le i \le n} S_n,$$

   voor $n = 1, 2, \cdots, N$. Show that the process $(M_1, S_1), \cdots (M_N, S_N)$ is Markov w.r.t. P.

Solution (a): Since $X_0, \cdots, X_N$ is a Markov process and $V_N = g_n(X_n)$ with $g_n(x) = x^2$, the result follows from Theorem 2.5.8, and which can be easily proved with backward induction as follows. From the hypothesis, the result is true for N. Assume it is true for $n \le N$, we will show it is true for $n-1$. Now $V_{n-1} = \widetilde{E}_n((1+r)^{-1}V_n)$, and by the induction hypothesis, $V_n = g_n(X_n)$ for some function g_n. Since the process, $X_0, \cdots, X_n$ is Markov w.r.t. $\widetilde{P}$, there exists a function h_{n-1} such that

$$\widetilde{E}_n(V_n) = \widetilde{E}_n(g_n(X_n)) = h_{n-1}(X_{n-1}).$$

Set $g_{n-1} = (1+r)^{-1}h_{n-1}$, we then have

$$V_{n-1} = (1+r)^{-1}\widetilde{E}_n(V_n) = g_{n-1}(X_{n-1}).$$

Solution (b): We first observe that $Z_{n+1} = Z_n + X_n U_{n+1}$, and $E_n(U_{n+1}) = E(U_{n+1}) = 0$. Thus,

$$E_n(U_{n+1}) = Z_n + X_n E_n(U_{n+1}) = Z_n.$$

So, $Z_0, Z_1, \ldots, Z_N$ is a martingale w.r.t. P.

Solution (c): First note that $M_{n+1} = \min(M_n, S_{n+1})$ and $S_{n+1} = S_n + U_{n+1}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any function, then

$$f(M_{n+1}, S_{n+1}) = f(\min(M_n, S_{n+1}), S_n + U_{n+1}) = F(M_n, S_n, U_{n+1}).$$

Since $M_n$and S_n depend on the first n tosses while U_{n+1} is independent of the first n tosses, we have by the independence Lemma that

$$E_n(f(M_{n+1}, S_{n+1})) = E_n(F(M_n, S_n, U_{n+1})) = g(M_n, S_n),$$

where

$$g(m,s) = E(F(m,s,U_{n+1})) = \frac{1}{2}\left(f(\min(m, s+\frac{1}{2}), s+\frac{1}{2}) + f(\min(m, s-\frac{1}{2}), s-\frac{1}{2})\right).$$

Hence, $(M_1, S_1), \cdots (M_N, S_N)$ is Markov w.r.t. P.