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Uitwerkingen Deeltentamen 1 Inleiding Financiele Wiskunde, 2011-12

* Punten per opgave:

opgave:	1	2	3
punten:	50	30	20

1. Consider a 2-period binomial model with $S_0 = 100$, $u = 1.2$, $d = 0.9$, and $r = 0.1$. Suppose the real probability measure P satisfies $P(H) = p = \dfrac{1}{2} = P(T)$.

   (a) Consider an option with payoff $V_2 = \max(S_1, S_2) - 100$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = HT$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ so that so that the corresponding wealth process satisfies $X_0 = V_0$ (your answer in part (a)) and $X_2(HT) = V_2(HT)$.

   (c) Suppose a trader is selling the above option for a price $T > V_0$. Explain how the trader can perform arbitrage, i.e. with begin wealth equals to zero he can build a portfolio that has at time 2 a non-negative value with probability 1.

   (d) Consider the utility function $U(x) = \sqrt{x}$ $(x > 0)$. Show that the random variable $X = X_2$ (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = X_0$ is given by

   $$X = X_2 = \frac{(1.1)^2 X_0}{Z^2 E(Z^{-1})},$$

   where Z is the Radon Nikodym derivative of $\widetilde{P}$ with respect to P.

   (e) Assume in part (e) that $X_0 = 100$. Determine the value of the optimal portfolio process $\{\Delta_0, \Delta_1\}$ and the value of the corresponding wealth process $\{X_0, X_1, X_2\}$.

Solution (a): We first calculate the risk-neutral probability measure $\widetilde{P}$, we have $\widetilde{P}(H) = \widetilde{p} = 2/3$ and $\widetilde{P}(T) = \widetilde{q} = 1/3$. We start with the value of V_2, we have $V_2(HH) = 44, V_2(HT) = 20, V_2(TH) = 8, V_2(TT) = 0$. Then

$$V_1(H) = \frac{1}{1.1}[\frac{2}{3}(44) + \frac{1}{3}(20)] = 32.73,$$

and

$$V_1(T) = \frac{1}{1.1}[\frac{2}{3}(8) + \frac{1}{3}(0)] = 4.85,$$

leading to

$$V_0 = \frac{1}{1.1}[\frac{2}{3}(32.72) + \frac{1}{3}(4.85)] = 21.31.$$

Solution (b): If $\omega_1\omega_2 = TH$, then

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{32.73 - 4.85}{120 - 90} = 0.93,$$

and

$$\Delta_1(T) = \frac{V_1(TH) - V_1(TT)}{S_1(TH) - S_1(TT)} = \frac{8 - 0}{108 - 81} = 0.296.$$

Leading to

$$X_1(T) = \Delta_0 S1_(T) + 1.1(V_0 - \Delta_0 S_0) = 4.85,$$

and

$$X_2(TH) = \Delta_1(T)S_2(TH) + 1.1(X_1(T) - \Delta_1(T)S_1(T)) = 8.$$

Solution (c):
–At time 0, sell the option for T euros and use V_0 to start a self financing portfolio which at time 2 has value equals the payoff of the option. Put the rest $V_0 - T$ in the bank.
–At time 2, your self-financing portfolio has value V_2 which you use to pay the payoff of the buyer of the option, and in the bank you have $(T - V_0)(1.1)^2 > 0$.

Solution (d): Notice that the function $U(x) = \sqrt{x}$, $x > 0$ is strict concave with $U'(x) = \frac{1}{2\sqrt{x}}$. We apply Theorem 3.3.6, we find that the inverse I of U' is given by $I(x) = \frac{1}{4x^2}$. Thus, the optimal solution is given by

$$X_2 = X = I\left(\frac{\lambda Z}{(1.1)^2}\right) = \frac{(1.1)^4}{4\lambda^2 Z^2},$$

and satisfies the constraint

$$X_0 = E\left(\frac{XZ}{(1.1)^2}\right) = \frac{(1.1)^2}{4\lambda^2}E\left(Z^{-1}\right).$$

Hence, $4\lambda^2 = \frac{(1.1)^2 E(Z^{-1})}{X_0}$, and

$$X = \frac{X_0(1.1)^2}{Z^2 E(Z^{-1})}.$$

Solution (e): To find the optimal portfolio and corresponding wealth processes, we first determine explicitly the the random variable $X = X_2$, and then we apply

Theorem 1.2.2 with $X_0 = 100$. We begin by find the Radon Nikodym derivative Z. We have

$$Z(HH) = \frac{16}{9},\ Z(HT) = Z(TH) = \frac{8}{9},\ Z(TT) = \frac{4}{9}.$$

Next, we find

$$E(Z^{-1}) = \frac{9}{16} \times \frac{1}{4} + \frac{9}{8} \times \frac{1}{4} + \frac{9}{8} \times \frac{1}{4} + \frac{9}{4} \times \frac{1}{4} = 1.27.$$

Thus,

$$X = X_2 = \frac{X_0(1.1)^2}{Z^2 E(Z^{-1})} = \frac{95.6}{Z^2}.$$

This leads to

$$X_2(HH) = 30.25,\ X_2(HT) = X_2(TH) = 120.99,\ X_2(TT) = 483.98.$$

Hence,

$$X_1(H) = \frac{1}{1.1}[\frac{2}{3}(30.25) + \frac{1}{3}(120.99)] = 55,$$
$$X_1(T) = \frac{1}{1.1}[\frac{2}{3}(120.99) + \frac{1}{3}(483.98)] = 220.$$

Notice that

$$X_0 = \frac{1}{1.1}[\frac{2}{3}(55) + \frac{1}{3}(220)] = 100$$

as required. The optimal portfolio is given by

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{55 - 220}{120 - 90} = -5.5,$$
$$\Delta_(H) = \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} = \frac{30.25 - 120.99}{144 - 108} = -2.52,$$
$$\Delta_1(T) = \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2T(T)} = \frac{120.99 - 483.98}{108 - 81} = -13.44.$$

2. Consider the N-period Binomial model with risk neutral probability measure $\widetilde{P}$. Suppose $X_0, X_1, \cdots, X_N$ is an adapted process satisfying $X_i > -1$ for all $i = 0, 1, \cdots, N$. Define a process $Y_0, Y_1, \cdots, Y_N$ by

$$Y_0 = 1, \quad \text{and } Y_n = \frac{1}{(1 + X_0)\cdots(1 + X_{n-1})},\ n = 1, \cdots, N.$$

   (a) Let $U_n = \widetilde{E}_n\left[\frac{Y_N}{Y_n}\right]$, $n = 0, 1, \cdots, N$. Show that the process $Y_0U_0, Y_1U_1, \cdots, Y_NU_N$ is a martingale with respect to $\widetilde{P}$.

   (b) Let $\Delta_0, \cdots, \Delta_{N-1}$ be an adapted process, and W_0 a fixed positive real number. Define for $n = 0, 1, \cdots, N - 1$,

$$W_{n+1} = \Delta_n U_{n+1} + (1 + X_n)(W_n - \Delta_n U_n).$$

   Show that the process

$$Y_0W_0, Y_1W_1, \cdots, Y_NW_N$$

   is a martingale with respect to $\widetilde{P}$.

(c) Let U_n be as given in part (a). Set $I_0 = 0$ and define $I_n = \sum_{j=0}^{n-1} Y_{j+1}(U_{j+1} - U_j)$, $n = 1, \cdots, N$. Show that $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\widetilde{P}$.

Solution (a): First note that the process $\{X_n : n = 0, \cdots, N\}$ is adapted, hence the random variable Y_n is known at time $n-1$, i.e. depends on the first $n-1$ tosses, $n = 1, \cdots, N$. Hence,

$$U_n = \widetilde{E}_n\left[\frac{Y_N}{Y_n}\right] = \frac{1}{Y_n}\widetilde{E}_n[Y_N],$$

which implies $Y_n U_n = \widetilde{E}_n[Y_N]$. Using the iteration property of conditional expectations, or directly Theorem 3.2.1, one has that the process $Y_0U_0, Y_1U_1, \cdots, Y_NU_N$ is a martingale with respect to $\widetilde{P}$.

Solution (b): It is clear that the process $W_0, \cdots, W_N$ is adapted. Since Y_n depends on the first $n-1$ tosses, we see that the process $Y_0W_0, Y_1W_1, \cdots, Y_NW_N$ is also adapted. Furthermore, $1 + X_n = \frac{Y_n}{Y_{n+1}}$. Thus,

$$\begin{aligned}
\widetilde{E}_n(W_{n+1}) &= \Delta_n \widetilde{E}_n(U_{n+1}) + (1 + X_n)(W_n - \Delta_n U_n) \\
&= \Delta_n \widetilde{E}_n(\widetilde{E}_{n+1}(\frac{Y_N}{Y_{n+1}})) + \frac{Y_n}{Y_{n+1}}(W_n - \Delta_n U_n) \\
&= \Delta_n \widetilde{E}_n(\frac{Y_N}{Y_{n+1}}) + \frac{Y_n}{Y_{n+1}}W_n - \Delta_n \frac{Y_n}{Y_{n+1}}\widetilde{E}_n(\frac{Y_N}{Y_n}) \\
&= \Delta_n \widetilde{E}_n(\frac{Y_N}{Y_{n+1}}) + \frac{Y_n}{Y_{n+1}}W_n - \Delta_n \widetilde{E}_n(\frac{Y_N}{Y_{n+1}}) \\
&= \frac{Y_n}{Y_{n+1}}W_n.
\end{aligned}$$

Thus, $Y_nW_n = Y_{n+1}\widetilde{E}_n(W_{n+1}) = \widetilde{E}_n(Y_{n+1}W_{n+1})$, and

$$Y_0W_0, Y_1W_1, \cdots, Y_NW_N$$

is a martingale with respect to $\widetilde{P}$.

Solution (c): First note that Y_{n+1} is known at time n, and

$$I_{n+1} = I_n + Y_{n+1}(U_{n+1} - U_n).$$

From part (a), we have that $U_0, \cdots, U_N$ is a martingale with respect to $\widetilde{P}$, and hence $\widetilde{E}_n(U_{n+1} - U_n) = 0$. Thus,

$$\widetilde{E}_n(I_{n+1}) = I_n + Y_{n+1}\,\widetilde{E}_n(U_{n+1} - U_n) = I_n.$$

Therefore, $I_0, I_1, \cdots, I_N$ is a martingale with respect to $\widetilde{P}$.

3. Consider the N-period binomial model, with expiration process N, up factor u, down factor d and interst rate r. Let $\widetilde{P}$ be the risk neutral probability and P the real probability. We denote by $p = P(H)$ and $\widetilde{p} = \widetilde{P}(H)$. Let $S_0, S_1, \cdots, S_N$ be the corresponding price process.

(a) Define $Y_n = \sum_{k=0}^{n} S_k$. Show that the process

$$(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$$

is Markov with respect to P and $\widetilde{P}$.

(b) Let $V_N = \left(S_N - \dfrac{Y_N}{N+1} \right)^+$. Show that for each $n = 0, 1, \cdots, N$, there exists a function f_n such that

$$E_n(ZV_N) = Z_n(1+r)^{N-n} f_n(Y_n, S_n),$$

where Z is the Radon-Nikodym derivative of $\tilde{P}$ with respect to P, and $Z_n = E_n(Z)$, $n = 0, 1, \cdots, N$.

Solution (a): Define $Z_{n+1} = \dfrac{S_{n+1}}{S_n}$ for $n = 0, 1, \cdots, N-1$. Note that Z_{n+1} is independent of the first n tosses, and

$$Y_{n+1} = Y_n + Z_{n+1} S_n, \text{ and } S_{n+1} = Z_{n+1} S_n.$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any function, by the Independence Lemma, we have

$$E_n\left(f(Y_{n+1}, S_{n+1})\right) = E_n\left(f(Y_n + Z_{n+1} S_n, Z_{n+1} S_n)\right) = g(Y_n, S_n),$$

where

$$g(y, s) = E(f(y + Z_{n+1} s, Z_{n+1} s)) = pf(y + us, us) + qf(y + ds, ds).$$

A similar calculation shows that

$$\widetilde{E}_n\left(f(Y_{n+1}, S_{n+1})\right) = \widetilde{E}_n\left(f(Y_n + Z_{n+1} S_n, Z_{n+1} S_n)\right) = h(Y_n, S_n),$$

where

$$h(y, s) = \widetilde{E}(f(y + Z_{n+1} s, Z_{n+1} s)) = \widetilde{p} f(y + us, us) + \widetilde{q} f(y + ds, ds).$$

Hence, the process

$$(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$$

is Markov with respect to P and $\widetilde{P}$.

Solution (b): Let $f(y, s) = (s - y(n+1)^{-1})^+$, then $V_N = f(Y_N, S_N)$. Since $(Y_0, S_0), (Y_1, S_1), \ldots, (Y_N, S_N)$ is Markov with respect to $\widetilde{P}$, by Theorem 2.5.8, for each $n = 0, 1, \cdots, N$, there exists a function f_n such that

$$V_n = \widetilde{E}(V_N (1+r)^{-(N-n)}) = f_n(Y_n, S_n),$$

(note that $f = f_N$). Thus, by Lemma 3.2.6

$$E_n(ZV_N) = Z_n \widetilde{E}_n(V_N) = Z_n \,(1+r)^{N-n} f_n(Y_n, S_n).$$