<center>**JUSTIFY YOUR ANSWERS!!**</center>

Please note:

- **Allowed: calculator, course-content material and notes *handwritten by you***

- **NO PHOTOCOPIED MATERIAL IS ALLOWED**

- **NO BOOK OR PRINTED MATERIAL IS ALLOWED**

- **If you use a result given as an exercise, you are expected to include (copy) its solution unless otherwise stated**

NOTE: The test consists of four questions for a total of 10.6 points plus a bonus problem worth 1.5 pts. The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Life insurance followed by perpetuity] A person subscribes a life-insurance plan for 30000 E that costs 200E per semester. At the end of each semester the person can decide either to continue with the plan or to collect the payments made so far (plus accrued interest). Assume the (yearly) interest remains constant and equal to 5%.

(a) (0.8 pts.) Determine at which moment the plan becomes uninteresting because the money to be collected exceeds the amount to be received by the inheritors in case of death.

(b) After 20 years the person decides to collect the money and use it to buy a perpetuity, that guarantees a yearly payment of C euros in principle forever. In case of death, however, the institution will pay to the inheritors the (discounted) amount of the (infinitely many) remaining payments.

-i- (0.4 pts.) Determine C.

-ii- (0.4 pts.) The person dies 20 years after purchasing the perpetuity. Show that the amount paid to the inheritors is exactly the same as the purchasing price of the perpetuity.

-iii- (0.2 pts.) Can you explain why after 20 years of payments the institution owes to the owner of the perpetuity the same amount as at the beginning?

Exercise 2. [Martingales and submartingales] A biased coin, with a probability p of showing head, is repeatedly tossed. Let $(\mathcal{F}_n)$ be the filtration of the binary model, in which $\mathcal{F}_n$ are the events determined by the first n tosses. A stochastic process (X_j) is defined such that

$$X_j = \begin{cases} 1 & \text{if } j\text{-th toss results in head} \\ -1 & \text{if } j\text{-th toss results in tail} \end{cases} \quad \text{for } j = 1, 2, \dots$$

Consider the process

$$M_0 = 0$$
$$M_n = \sum_{j=1}^{n} X_j .$$

(a) (0.8 pts.) Determine the values p for which (M_n) (i) a martingale, (ii) a sub-martingale and (iii) a super-martingale adapted to the filtration $(\mathcal{F}_n)$.

(b) (0.4 pts.) Show that for $p \geq 1/2$ the process (M_n^2) is a sub-martingale.

Exercise 3. [(Non-) stopping times] Consider the probability setup of a three-period binary model, namely
$$\Omega = \Omega_3 = \big\{(\omega_1, \omega_2, \omega_3) : \omega_i = H, T\big\}$$
and the well known filtration $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$. Consider the following two strategies to stop:

$$\begin{aligned}
\tau_1 &= \text{Second time an ``H" shows up} \\
\tau_2 &= \text{One-before-last time a ``T" shows up.}
\end{aligned}$$

Show

(a) (0.8 pts.) τ_1 is a stopping time.

(b) (0.8 pts.) τ_2 is *not* a stopping time.

Exercise 4. [Straddles] Consider a stock with initial price $S_0 = 4$ following a binomial model with $u = 2$ and $d = 1/2$. That is, at the end of each period, the price can either double or be halved. Bank interest is 10% for each period.

(a) (0.5 pts.) Determine the risk-neutral probability.

(b) An investor buys a two-period American option with intrinsic value

$$G_n = |S_n - K|_+ + |K - S_n|_+ \qquad n = 0, 1, 2 .$$

This is a combination of an American put and an American call options with strike value K. Consider

$$K =; \ S_0 \, 1.1^2 = 4.84 \tag{1}$$

(same price as what s/he would get if depositing the initial stock value in the bank).

-i- Compute
   - (1 pt.) The value process V_n, $n = 0, 1, 2$ of the option.
   - (0.5 pts.) The consumption process C_n, $n = 0, 1, 2$.
   - (0.5 pts.) The hedging policy Δ_n, $n = 0, 1$ for the issuer.
-ii- (1 pt.) Determine the optimal exercise time τ^* for the owner.
-iii- (0.5 pts.) Verify the identity
$$V_0 = \widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau^* \leq N\}} \overline{G}_{\tau^*}\right] .$$

-iv- (0.5 pts.) Show that the discounted process $\overline{V}_n$, $n = 0, 1, 2$ is *not* a martingale.
-v- (0.5 pts.) Show that, nevertheless, the stopped discounted process $\overline{V}_n^{\tau^*}$, $n = 0, 1, 2$ is a martingale.

(c) The investor is also offered the European version of the option, that is a two-period European option with

$$V_2 \;=\; |S_2 - K|_+ + |K - S_2|_+ \;.$$

-i- (0.7 pts.) Show that its purchasing price satisfies

$$V_0^{\mathrm{Eu}} \;=\; V_0^{\mathrm{Eu,call}} + V_0^{\mathrm{Eu,put}}$$

where $V_0^{\mathrm{Eu,call}}$ and $V_0^{\mathrm{Eu,put}}$ are, respectively, the prices of European call and put options with strike value K.

-ii- (0.3 pts.) Invoking a result seeing as an exercise (no need to copy the solution of the exercise), show that, if K is given by (1),

$$V_0^{\mathrm{Eu}} \;=\; 2\, V_0^{\mathrm{Eu,call}}$$

Bonus problem

Bonus. [**Black-Schole-Merton put option**] Consider a BSM market with initial stock price S_0, (continuously compounded) risk-free yearly interest rate r, expected return μ per annum and volatility σ per annum. Show that the fair price of an European put option with strike price K with maturity time T (in years) is

$$V_0^{\mathrm{put}} \;=\; -S_0\,\mathcal{N}(-L_+) + K\,\mathrm{e}^{-rT}\,\mathcal{N}(-L_-)$$

with

$$L_\pm \;=\; \frac{1}{\sigma\sqrt{T}}\Big[\log(S_0/K) + rT \pm \sigma^2 T/2\Big]\;.$$