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Final Exam: Inleiding Financiele Wiskunde 2018-2019

(1) Consider a Brownian motion $\{W(t) : t \geq 0\}$ with filtration $\{\mathcal{F}(t) : t \geq 0\}$. Suppose that the price process $\{S(t) : t \geq 0\}$ of a certain stock is modelled as the following Itô-process

$$S(t) = S(0) + \int_0^t \mu S(u)\, du + \int_0^t \sigma\, dW(u).$$

(a) Use Itô-Doeblin formula to show that $e^{-\mu t} S(t) = S(0) + \int_0^t e^{-\mu u} \sigma\, dW(u)$. (1 pt)

(b) Determine the distribution of $S(t)$ and calculate $\mathbb{P}(S(t) < 0)$ for $t > 0$. (1 pt)

(2) Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider two price processes $\{S_1(t) : t \geq 0\}$ and $\{S_2(t) : t \geq 0\}$ with corresponding SDE given by

$$dS_1(t) = \alpha S_1(t)\, dW_1(t) + \beta S_1(t)\, dW_2(t)$$
$$dS_2(t) = \gamma S_2(t)\, dt + \sigma S_2(t)\, dW_1(t),$$

where $\alpha, \beta, \gamma, \sigma$ are positive constants.

(a) Show that $\{S_1(t) S_2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (1 pt)

(b) Show that $\mathbb{E}[S_1(t) S_2(t)] = S_1(0) S_2(0) e^{(\gamma + \alpha\sigma)t}$, $t \geq 0$. (You are allowed to interchange integrals and expectations). (1 pt)

(c) Consider a finite time T (expiration date), and suppose the interest rate is a constant, i.e. $R(t) = r$ for all $t > 0$. Show that the market price equations have a unique solution, and determine the risk-neutral probability measure $\widetilde{\mathbb{P}}$ for the process $\{(S_1(t), S_2(t) : 0 \leq t \leq T\}$. (1.5 pt)

(3) Let T fe finite horizon and let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mu)$ with filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$, where $\mathcal{F}(T) = \mathcal{F}$. Suppose that the price process $\{S(t) : 0 \leq t \leq T\}$ of a certain stock is given by

$$S(t) = \exp\left\{2W(t) + \frac{t^2}{2} - 2t\right\}$$

(a) Show that $\{S(t) : 0 \leq t \leq T\}$ is an Itô-process. (1 pt)

(b) Let r be a constant interest rate. Find a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the discounted process $\{e^{-rt} S(t) : 0 \leq t \leq T\}$ is a martingale under $\widetilde{\mathbb{P}}$. (1 pt)

(4) Consider a Brownian motion $\{W(t) : t \geq 0\}$ with the natural filtration $\{\mathcal{F}(t) : t \geq 0\}$, where $\mathcal{F}(t) = \sigma(\{W(s) : s \leq t\})$. Consider the stochastic process $\{M(t) : t \geq 0\}$, with

$$M(t) = \left(\int_0^t s W^2(s)\, dW(s)\right)^2 - \int_0^t s^2 W^4(s)\, ds.$$

(a) Determine the value of $\mathbb{E}[M(t)]$ for $t \geq 0$. (1 pt)

(b) Prove that the stochastic process $\{M(t) : t \geq 0\}$ is a martingale with respect to the natural filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1.5 pt)