Solutions Mid-Term: Inleiding Financiele Wiskunde 2019-2020

(1) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise independent sets in $\mathcal{F}$ (i.e. $\mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n)\mathbb{P}(A_m)$ for $n \neq m$) satisfying $\mathbb{P}(A_n) = 1/2$ for all $n \geq 1$. Let $\mathbb{I}_{A_n}$ be the indicator function of the set A_n and $\sigma(\mathbb{I}_{A_n})$ the σ-algebra generated by the random variable $\mathbb{I}_{A_n}$, $n \geq 1$.

(a) Prove that $\sigma(\mathbb{I}_{A_n}) = \{\emptyset, \Omega, A_n, A_n^c\}$ and that the σ-algebras $\sigma(\mathbb{I}_{A_n})$ and $\sigma(\mathbb{I}_{A_m})$ are independent whenever $n \neq m$, i.e. $\mathbb{P}(C \cap D) = \mathbb{P}(C)\mathbb{P}(D)$ for any $C \in \sigma(\mathbb{I}_{A_n})$ and any $D \in \sigma(\mathbb{I}_{A_m})$. Conclude that $\mathbb{I}_{A_1}, \mathbb{I}_{A_2}, \cdots$ is a **pairwise independent** sequence. (1.5 pts)

(b) For $n \geq 1$, define $X_n = 2\mathbb{I}_{A_n} - 1$. Set $M_0 = 0$, $M_n = \sum_{k=1}^{n} 2^{k-1}X_k$ for $n \geq 1$ and let $Y_n = M_n^2 - \frac{(4^n-1)}{3}$ for $n \geq 0$. Consider the filtration $\{\mathcal{F}(n) : n \geq 0\}$ where $\mathcal{F}(0) = \{\emptyset, \Omega\}$ and $\mathcal{F}(n) = \sigma(\mathbb{I}_{A_1}, \cdots, \mathbb{I}_{A_n})$ = the smallest σ-algebra containing all sets of the form $\{\mathbb{I}_{A_j} \in B\}$ for any Borel set B and any $1 \leq j \leq n$. Prove that the process $\{Y_n : n \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(n) : n \geq 0\}$. (1.5 pts)

Proof(a): By definition $\sigma(\mathbb{I}_{A_n}) = \Big\{\{\mathbb{I}_{A_n} \in B\} : B \text{ is a Borel set}\Big\}$. Since $\mathbb{I}_{A_n}$ takes only the values 0 and 1, we see that

$$\{\mathbb{I}_{A_n} \in B\} := \begin{cases} \emptyset & \text{if } 0, 1 \notin B \\ A_n^c & \text{if } 0 \in B, \text{ and } 1 \notin B \\ A_n & \text{if } 1 \in B, \text{ and } 0 \notin B \\ \Omega & \text{if } 0, 1 \in B. \end{cases}$$

Thus, $\sigma(\mathbb{I}_{A_n}) = \{\emptyset, \Omega, A_n, A_n^c\}$.

Next wee need to show that the σ-algebras $\sigma(\mathbb{I}_{A_n})$ and $\sigma(\mathbb{I}_{A_m})$ are independent whenever $n \neq m$, i.e. $\mathbb{P}(C \cap D) = \mathbb{P}(C)\mathbb{P}(D)$ for any $C \in \sigma(\mathbb{I}_{A_n})$ and any $D \in \sigma(\mathbb{I}_{A_m})$. First note that $\sigma(\mathbb{I}_{A_n}) = \{\emptyset, \Omega, A_n, A_n^c\}$ and $\sigma(\mathbb{I}_{A_m}) = \{\emptyset, \Omega, A_m, A_m^c\}$. If C or D is either $\emptyset$ or Ω, then the result is trivially true. So we only need to consider the case $C \in \{A_n, A_n^c\}$ and $D \in \{A_m, A_m^c\}$. By hypothesis, $\mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n)\mathbb{P}(A_m)$. For the other cases, we first note that

$$\mathbb{P}(A_n) = \mathbb{P}(A_n \cap A_m^c) + \mathbb{P}(A_n \cap A_m) = \mathbb{P}(A_n \cap A_m^c) + \mathbb{P}(A_n)\mathbb{P}(A_m),$$

implying

$$\mathbb{P}(A_n \cap A_m^c) = \mathbb{P}(A_n) - \mathbb{P}(A_n)\mathbb{P}(A_m) = \mathbb{P}(A_n)\Big(1 - \mathbb{P}(A_m)\Big) = \mathbb{P}(A_n)\mathbb{P}(A_m^c).$$

Similarly,

$$\mathbb{P}(A_m) = \mathbb{P}(A_m \cap A_n^c) + \mathbb{P}(A_m \cap A_n) = \mathbb{P}(A_m \cap A_n^c) + \mathbb{P}(A_n)\mathbb{P}(A_m),$$

leading to $\mathbb{P}(A_m \cap A_n^c) = \mathbb{P}(A_m)\mathbb{P}(A_n^c)$. Finally,

$$\begin{aligned} \mathbb{P}\Big(A_n^c \cap A_m^c\Big) = \mathbb{P}\Big((A_n \cup A_m)^c\Big) &= 1 - \mathbb{P}\Big(A_n \cup A_m\Big) \\ &= 1 - (P(A_n) + \mathbb{P}(A_m) - \mathbb{P}(A_n \cap A_m)) \\ &= 1 - (P(A_n) + \mathbb{P}(A_m) - \mathbb{P}(A_n)\mathbb{P}(A_m)) \\ &= (1 - \mathbb{P}(A_n))(1 - \mathbb{P}(A_m)) \\ &= \mathbb{P}(A_n^c)\mathbb{P}(A_m^c). \end{aligned}$$

This shows that $\sigma(\mathbb{I}_{A_n})$ and $\sigma(\mathbb{I}_{A_m})$ are independent whenever $n \neq m$. Since by definition two random variables X and Y are independent if $\sigma(X)$ and $\sigma(Y)$ are independent, we conclude that the sequence $\mathbb{I}_{A_1}, \mathbb{I}_{A_2}, \cdots$ is pairwise independent.

Proof(b): First note that

$$X_n(\omega) = \begin{cases} 1 & \omega \in A_n \\ -1 & \omega \notin A_n. \end{cases}$$

From here we see that $\mathcal{F}(n) = \sigma(X_1, \cdots, X_n)$ and $\mathbb{E}(X_n) = 2\mathbb{P}(A_n) - 1 = 0$ for all $n \geq 1$. Since $\mathcal{F}(1) \subset \mathcal{F}(2) \subset \cdots \subset \mathcal{F}(n)$, we see that M_n is $\mathcal{F}(n)$-measurable implying that Y_n is $\mathcal{F}(n)$-measurable and hence the process $\{Y_n : n \geq 0\}$ is adapted to the filtration $\{\mathcal{F}(n) : n \geq 0\}$. To show that the process $\{Y_n : n \geq 0\}$ is a martingale, it is enough to show that $\mathbb{E}[Y_{n+1}|\mathcal{F}(n)] = Y_n$, for then by the repeated application of the iterated conditioning property we will have $\mathbb{E}[Y_n|\mathcal{F}(m)] = Y_m$ for any $m < n$ (see the solutions of the Mock Mid-term). Note that

$$M_{n+1}^2 = \Big(M_n + 2^n X_{n+1}\Big)^2 = M_n^2 + 2^{n+1} M_n X_{n+1} + 4^n X_{n+1}^2.$$

Since X_{n+1} is independent of $\mathcal{F}(n)$, we have $E[X_{n+1}|\mathcal{F}(n)] = \mathbb{E}[X_{n+1}] = 0$ and $E[X_{n+1}^2|\mathcal{F}(n)] = \mathbb{E}[X_{n+1}^2] = 1$. By linearity of the conditional expectation, the $\mathcal{F}(n)$-measurability of M_n and the *take out what you know* property, we have

$$\mathbb{E}[M_{n+1}^2|\mathcal{F}(n)] = M_n^2 + 2^{n+1} M_n \mathbb{E}[X_{n+1}] + 4^n \mathbb{E}[X_{n+1}^2] = M_n^2 + 4^n.$$

Thus,

$$\mathbb{E}[Y_{n+1}|\mathcal{F}(n)] = \mathbb{E}[M_{n+1}^2|\mathcal{F}(n)] - \frac{(4^{n+1}-1)}{3} = M_n^2 + 4^n - \frac{(4^{n+1}-1)}{3} = M_n^2 - \frac{(4^n - 1)}{3} = Y_n.$$

Therefore, $\{Y_n : n \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(n) : n \geq 0\}$.

(2) Let $\{W(t) : t \geq 0\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : t \geq 0\}$ be a filtration for the Brownian motion. Define a process $\{X(t) : t \geq 0\}$ by $X(t) = e^{tW(t) - t^3 + 1}$, $t \geq 0$.

(a) Prove that $\mathbb{P}(X(1) > 1) = 1/2$. (1 pt)

(b) Derive an expression for $\text{Var}[X(t)]$, the variance of $X(t)$. (1.5 pts)

(c) For $s < t$, determine an expression for $\mathbb{E}[X(t)|\mathcal{F}(s)]$. (1.5 pts)

Proof (a): We have $X(1) = e^{W(1)}$, with $W(1)$ a standard normal random variable (so mean zero and variance 1). Thus,

$$\begin{aligned} \mathbb{P}(X(1) > 1) &= \mathbb{P}(\ln(X(1) > 0) \\ &= \mathbb{P}(W(1) > 0) \\ &= 1 - \mathbb{P}(W(1) \leq 0) \\ &= 1 - N(0) = 1/2, \end{aligned}$$

where N denotes the cumulative distribution function of the standard normal distribution.

Proof (b): We first calculate the expectation of $X(t)$, we have

$$\mathbb{E}[X(t)] = e^{-t^3+1}\mathbb{E}[e^{tW(t)}] = e^{-t^3+1} e^{\frac{1}{2}t^3} = e^{-\frac{1}{2}t^3+1},$$

where in the second equality we used that the moment generating function of the $\mathcal{N}(0,t)$ random variable $W(t)$ has value $\mathbb{E}[e^{uW(t)}] = e^{\frac{1}{2}u^2 t}$ (in our case $u = t$). Next we calculate the expectation of $X^2(t) = e^{2tW(t) - 2t^3 + 2}$,

$$\mathbb{E}[X^2(t)] = e^{-2t^3+2}\mathbb{E}[e^{2tW(t)}] = e^{-2t^3+2} e^{\frac{1}{2}4t^3} = e^2.$$

Thus,

$$\text{Var}(X(t) = \mathbb{E}[X^2(t)] - (\mathbb{E}[X(t)])^2 = e^2 - e^{-t^3+2} = e^2(1 - e^{-t^3}).$$

Proof (c): Using the fact that $W(s)$ is $\mathcal{F}(s)$-measurable and that $W(t)-W(s)$ is independent of $\mathcal{F}(s)$, we have by the properties of conditional expectation,

$$\begin{aligned}\mathbb{E}[X(t)|\mathcal{F}(s)] &= e^{-t^3+1}\mathbb{E}[e^{tW(t)}|\mathcal{F}(s)]\\ &= e^{-t^3+1}\mathbb{E}[e^{t(W(t)-W(s))+tW(s)}|\mathcal{F}(s)]\\ &= e^{-t^3+1}e^{tW(s)}\mathbb{E}[e^{t(W(t)-W(s))}|\mathcal{F}(s)]\\ &= e^{-t^3+1}e^{tW(s)}\mathbb{E}[e^{t(W(t)-W(s))}]\\ &= e^{-t^3+1}e^{tW(s)}e^{\frac{1}{2}t^2(t-s)}\\ &= e^{tW(s)-\frac{1}{2}t^2(t+s)+1},\end{aligned}$$

where in the first equality we used the linearity of the conditional expectation, in the third equality we used the property *take out what you know*, in the fourth equality we used the independence of $W(t)-W(s)$ and $\mathcal{F}(s)$ and in the fifth equality we used the fact that the moment generating function of $W(t)-W(s)$ is given by $\mathbb{E}[e^{u(W(t)-W(s))}] = e^{\frac{1}{2}u^2(t-s)}$ for $u \in \mathbb{R}$.

(3) Let $\{W(t) : t \geq 0\}$ and $\{V(t) : t \geq 0\}$ be two **independent** Brownian motions defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By independence we mean that $W(t)$ and $V(s)$ are independent for all $s, t > 0$. Let $0 < \rho < 1$ be a positive real number and define a process $\{Z(t) : t \geq 0\}$ by $Z(t) = \rho\, W(t) + \sqrt{1-\rho^2}\, V(t)$. Prove that the process $\{Z(t) : t \geq 0\}$ is a Brownian motion. (3 pts)

(**Hint**: if X and Y are independent normally distributed random variables with X being $\mathcal{N}(\mu_1, \sigma_1^2)$ and Y being $\mathcal{N}(\mu_2, \sigma_2^2)$, then $X+Y$ is normally $\mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ distributed).

Proof: We check that the process $\{Z(t) : t \geq 0\}$ satisfies all the properties of a Brownian motion. We have

(i) $Z(0) = \rho\, W(0) + \sqrt{1-\rho^2}\, V(0) = 0$.

(ii) Since both $\{W(t) : t \geq 0\}$ and $\{V(t) : t \geq 0\}$ have continuous paths and a linear combination of continuous functions is continuous, we that the process has continuous paths.

(iii) Let $0 = t_0 < t_1 < \cdots < t_m$, then $W(t_{i+1}) - W(t_i)$ is independent of $W(t_{j+1}) - W(t_j)$ and $V(t_{i+1}) - V(t_i)$ is independent of $V(t_{j+1}) - V(t_j)$ for all $i \neq j$. Furthermore, $W(t_{i+1}) - W(t_i)$ is independent of $V(t_{j+1}) - V(t_j)$ for all $i, j = 1, \cdots m$. Thus the increments $Z(t_1) - Z(t_0), \cdots, Z(t_m) - Z(t_{m-1})$ are independent.

(iv) Let $s < t$, then $Z(t) - Z(s) = \rho(W(t) - W(s)) + \sqrt{1-\rho^2}\,(V(t) - V(s))$. By hypothesis the random variables $W(t) - W(s)$ and $V(t) - V(s)$ are independent and both are normally $\mathcal{N}(0, t-s)$ distributed. Thus, $\rho\,(W(t) - W(s))$ and $\sqrt{1-\rho^2}\,(V(t) - V(s))$ are independent with $\rho\,(W(t) - W(s))$ normally $\mathcal{N}(0, \rho^2(t-s))$ distributed and $\sqrt{1-\rho^2}\,(V(t) - V(s))$ normally $\mathcal{N}(0, (1-\rho^2)(t-s))$ distributed. Using the hint we have that $Z(t) - Z(s)$ is normally $\mathcal{N}(0, t-s)$.

Therefore, $\{Z(t) : t \geq 0\}$ is a Brownian motion.