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Solutions Retake Exam: Inleiding Financiele Wiskunde 2019-2020

(1) Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Let $c(t)$ be a deterministic function with $c(0) = 0$. Let r be a given interest rate and consider the price process $\{S(t) : 0 \leq t \leq T\}$ given by $S(t) = e^{W(t)+c(t)}$.

(a) Find an expression for $c(t)$ under which the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale with respect to the probability measure $\mathbb{P}$. (1.5 pts)

(b) Consider the expression obtained in part (a), i.e. assume that the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under the probability measure $\mathbb{P}$. Consider a financial derivative with payoff at time T given by $V(T) = \mathbb{I}_{\{S(T)>K\}}$ (i.e. $V(T)$ has value 1 if $S(T) > K$ and 0 otherwise), here K is some given positive constant. Find the *fair* price of this option at time 0. (1.5 pts)

Proof (a): For the process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ to be a martingale, it should hold that $\mathbb{E}[e^{-rt}S(t)] = S(0) = 1$ for all $0 \leq t \leq T$. This gives

$$1 = e^0 = \mathbb{E}[e^{-rt}S(t)] = e^{-rt+c(t)}\mathbb{E}[e^{W(t)}] = e^{-rt+c(t)+\frac{1}{2}t}.$$

Equating the exponents, we get $c(t) = rt - \frac{1}{2}t = t(r - \frac{1}{2})$. We now check that this expression for $c(t)$ indeed gives us a martingale. So let $s < t \leq T$, since $W(s)$ is $\mathcal{F}(s)$ measurable and $W(t) - W(s)$ is independent of $\mathcal{F}(s)$, we have

$$\begin{aligned}
\mathbb{E}[e^{-rt}S(t)|\mathcal{F}(s)] &= \mathbb{E}[e^{-rt}e^{W(t)+t(r-\frac{1}{2})}|\mathcal{F}(s)] \\
&= \mathbb{E}[e^{W(t)-\frac{1}{2}t}|\mathcal{F}(s)] \\
&= e^{-\frac{1}{2}t}\mathbb{E}[e^{(W(t)-W(s))+W(s)}|\mathcal{F}(s)] \\
&= e^{-\frac{1}{2}t+W(s)}\mathbb{E}[e^{(W(t)-W(s))}] \\
&= e^{-\frac{1}{2}t+W(s)}e^{\frac{1}{2}(t-s)} \\
&= e^{-\frac{1}{2}s+W(s)} \\
&= e^{-rs}e^{(r-\frac{1}{2})s+W(s)} \\
&= e^{-rs}S(s).
\end{aligned}$$

Therefore, with $c(t) = (r - \frac{1}{2})t$, the process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale with respect to the probability measure $\mathbb{P}$.

Proof (b): From part (a), with $c(t) = t(r - \frac{1}{2})$ the discounted price process is a martingale under $\mathbb{P}$. Hence $\mathbb{P}$ is a risk-neutral measure and the fair price of the option is given by

$$\begin{aligned}
V(0) &= \mathbb{E}[e^{-rT}V(T)] \\
&= e^{-rT}\mathbb{E}[\mathbb{I}_{\{S(T)>K\}}] \\
&= e^{-rT}\mathbb{P}(S(T) > K) \\
&= e^{-rT}\mathbb{P}\Big(e^{W(T)+T(r-\frac{1}{2})} > K\Big) \\
&= e^{-rT}\mathbb{P}\Big(W(T) > \ln K - T(r - \frac{1}{2})\Big) \\
&= e^{-rT}\mathbb{P}\Big(\frac{W(T)}{\sqrt{T}} > \frac{\ln K - T(r - \frac{1}{2})}{\sqrt{T}}\Big) \\
&= e^{-rT}\Big(1 - N\Big(\frac{\ln K - T(r - \frac{1}{2})}{\sqrt{T}}\Big)\Big),
\end{aligned}$$

where $N(y)$ stands for the standard normal cumulative distribution function and we have used the fact that $\frac{W(T)}{\sqrt{T}}$ is standard normally distributed.

(2) Let $\{W_1(t), W_2(t)) : t \geq 0\}$ be a two dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the two processes $\{Z(t) : t \geq 0\}$ and $\{B(t) : t \geq 0\}$ defined by

$$Z(t) = 1 + e^{-W_1(t)} \int_0^t e^{W_1(u)} dW_2(u)$$

and

$$B(t) = \int_0^t \frac{1}{\sqrt{1+Z^2(u)}}\, dW_1(u) - \int_0^t \frac{Z(u)}{\sqrt{1+Z^2(u)}}\, dW_2(u).$$

(a) Use Lévy's characterization to prove that the process $\{B(t) : t \geq 0\}$ is a one dimensional Brownian motion. (1 pt)

(b) Prove that the process $\{Z(t) : t \geq 0\}$ can be written as

$$Z(t) = 1 + W_2(t) - \int_0^t Z(u)\, dW_1(u) + \frac{1}{2}\int_0^t Z(u)\, ds.$$

(1.5 pts)

(c) Prove that $\mathbb{E}[Z(t)] = e^{\frac{1}{2}t}$, for $t \geq 0$. (1 pt)

Proof (a): We will use Lévy's characterization of a Brownian motion. Clearly $B(0) = 0$ and since Itô integrals have continuous paths and are martingales, we see that $\{B(t) : t \geq 0\}$ has continuous paths and is a sum of two martingales hence also a martingale. It remains to show that $[B, B](t) = t$. Note that

$$dB(t) = \frac{1}{\sqrt{1+Z^2(t)}}\, dW_1(t) - \frac{Z(t)}{\sqrt{1+Z^2(t)}}\, dW_2(t),$$

thus

$$dB(t)dB(t) = \frac{1}{1+Z^2(t)} dt + \frac{Z^2(t)}{1+Z^2(t)} dt = dt.$$

Therefore, $[B, B](t) = t$ and by Lévy's characterization, $\{B(t) : t \geq 0\}$ is a one dimensional Brownian motion.

Proof (b): Let $X(t) = e^{-W_1(t)}$ and $Y(t) = \int_0^t e^{W_1(u)} dW_2(u)$, then $Z(t) = 1 + X(t)Y(t)$. By definition we have $dY(t) = e^{W_1(t)} dW_2(t)$. We will now derive the SDE for the process $\{X(t) : t \geq 0\}$ using Itô-Doeblin applied to the function $f(x) = e^{-x}$. We have $f_x(x) = -f(x)$ and $f_{xx}(x) = f(x)$, thus

$$dX(t) = df(W_1(t)) = -X(t)\, dW_1(t) + \frac{1}{2}X(t)\, dt,$$

that is

$$X(t) = 1 + \frac{1}{2}\int_0^t X(u)\, du - \int_0^t X(u)\, dW_1(u).$$

Since $W_1(t)$ and $W_2(t)$ are independent, it is easy to see that $dX(t)dY(t) = 0$. Applying Itô product rule we have,

$$\begin{aligned} dZ(t) = d(1 + X(t)Y(t)) &= X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \\ &= X(t)\Big(e^{W_1(t)} dW_2(t)\Big) + Y(t)\Big(- X(t)\, dW_1(t) + \frac{1}{2}X(t)\, dt\Big) \\ &= dW_2(t) - X(t)Y(t)\, dW_1(t) + \frac{1}{2}X(t)Y(t)\, dt \\ &= dW_2(t) - Z(t)\, dW_1(t) + \frac{1}{2}Z(t)\, dt. \end{aligned}$$

Since $Z(0) = 1$, the above shows that

$$Z(t) = 1 + W_2(t) - \int_0^t Z(u)\, dW_1(u) + \int_0^t \frac{1}{2} Z(u)\, du.$$

Proof (c): Since $\mathbb{E}[W_2(t)] = 0$ and $\mathbb{E}\Big[\int_0^t Z(u)\, dW_1(u)\Big] = 0$, we have by linearity of the expectation that

$$\mathbb{E}[Z(t)] = 1 + \mathbb{E}\Big[\int_0^t \frac{1}{2} Z(u)\, du\Big] = 1 + \frac{1}{2}\int_0^t \mathbb{E}[Z(u)]\, du,$$

where the second equality follows from Fubini (in fact Tonelli) since $Z(u) \geq 1$ (i.e. is non-negative). If we set $m(t) = \mathbb{E}[Z(t)]$, then the above equation reads $m(t) = 1 + \frac{1}{2}\int_0^t m(u)\, du$ and in differential form $\frac{dm(t)}{dt} = \frac{1}{2} m(t)$. This has solution $m(t) = m(0)e^{\frac{1}{2}}$. Since $m(0) = \mathbb{E}[Z(0)] = 1$, we have $\mathbb{E}[Z(t)] = e^{\frac{1}{2}t}$ as required.

(3) Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$ with

$$S(t) = S(0) \exp\Big\{ \int_0^t e^{-u}\, dW(u) + \int_0^t (1 - \frac{1}{2} e^{-2u})\, du \Big\}.$$

(a) Let $X(t) = \int_0^t e^{-u}\, dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2u})\, du$. Determine the distribution of $X(t)$. (1 pt)

(b) Prove that $\{S(t) : t \geq 0\}$ is an Itô process. (1 pt)

(c) Let r be a constant interest rate. Find the risk-neutral measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e. $\widetilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0$, $A \in \mathcal{F}$) such that the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\widetilde{\mathbb{P}}$. (1.5 pts)

Proof (a) : Let $Y(t) = \int_0^t e^{-u}\, dW(u)$. Since $Y(t)$ is the Itô integral of a deterministic process, by Theorem 4.4.9 $Y(t)$ is normally distributed with $\mathbb{E}[Y(t)] = 0$ and and $\mathrm{Var}[Y(t)] = \int_0^t e^{-2u}\, du = \frac{1}{2}(1 - e^{2t})$. Since $X(t) = Y(t) + \int_0^t (1 - \frac{1}{2}e^{-2u})\, du = Y(t) + t + \frac{1}{4}(e^{-2t} - 1)$, we see that $X(t)$ is **normally** distributed with mean $\mathbb{E}[X(t)] = t + \frac{1}{4}(e^{-2t} - 1)$ and variance $\mathrm{Var}[X(t)] = \mathrm{Var}[Y(t)] = \frac{1}{2}(1 - e^{2t})$.

Proof (b) : With $X(t) = \int_0^t e^{-u}\, dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2t})\, dt$ we have $dX(t) = e^{-t}\, dW(t) + (1 - \frac{1}{2}e^{-2t})\, dt$ and $dX(t)dX(t) = e^{-2t}\, dt$. Note that $S(t) = S(0)e^{X(t)}$, so let $f(x) = S(0)e^x$, then $f_x(x) = f_{xx}(x) = f(x)$. By Itô Doeblin we have,

$$\begin{aligned} dS(t) = df(X(t)) &= S(t)\, dX(t) + \frac{1}{2} S(t)\, dX(t)dX(t) \\ &= S(t)\Big(e^{-t}\, dW(t) + (1 - \frac{1}{2}e^{-2t})\, dt\Big) + \frac{1}{2} S(t) e^{-2t}\, dt \\ &= S(t)\, dt + S(t)e^{-t}\, dW(t). \end{aligned}$$

This shows that $S(t) = S(0) + \displaystyle\int_0^t S(u)\, du + \int_0^t S(u)e^{-u}\, dW(u)$, hence $\{S(t) : t \geq 0\}$ is an Itô process.

Proof (c) : Define $\theta(t) = \frac{1-r}{e^{-t}} = e^t(1 - r)$. Consider the random variable Z defined by

$$\begin{aligned} Z &= \exp\Big(- \int_0^T \theta(u)\, dW(u) - \frac{1}{2}\int_0^T \theta^2(u)\, du\Big) \\ &= \exp\Big(- \int_0^T e^t(1 - r)\, dW(u) - \frac{1}{2}\int_0^T e^{2u}(1 - r)\, du\Big). \end{aligned}$$

Note that $\int_0^t \theta(u)\, dW(u)$, $\int_0^t \theta^2(u)\, du$ and θ are continuous functions on the compact interval $[0, T]$, hence they are all bounded. This implies that $\mathbb{E}\Big[\int_0^T \theta^2(u)Z^2(u)\, du\Big] < \infty$. Define the measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}$ by $\widetilde{\mathbb{P}}(A) = \int_A Z\, d\mathbb{P}$ and consider the process $\{\widetilde{W}(t) : 0 \le t \le T\}$ with $\widetilde{W}(t) = \int_0^t \theta(u)\, du + W(t) = \int_0^t e^u(1-r)\, du + W(t) = (1-r)(e^t - 1) + W(t)$. By Girsanov's Theorem, the process $\{\widetilde{W}(t) : 0 \le t \le T\}$ is a Brownian motion under $\widetilde{\mathbb{P}}$ and hence is a martingale under $\widetilde{\mathbb{P}}$. Using the SDE obtained in part (a), together with Itô product rule, we have

$$\begin{aligned} d(e^{-rt}S(t)) &= e^{-rt}\, dS(t) - re^{-rt}S(t)\, dt \\ &= e^{-rt}\Big(S(t)\, dt + S(t)e^{-t}\, dW(t)\Big) - re^{-rt}S(t)\, dt \\ &= e^{-rt}S(t)\Big((1-r)\, dt + e^{-t}\, dW(t)\Big) \\ &= e^{-rt}S(t)\Big(e^{-t}\theta(t)\, dt + e^{-t}\, dW(t)\Big) \\ &= e^{-t(r+1)}S(t)\, d\widetilde{W}(t). \end{aligned}$$

Since $e^{-rt}S(t)$ is an Itô integral, we see that the discounted price process is a martingale under $\widetilde{\mathbb{P}}$.