Universiteit Utrecht Boedapestlaan 6
Mathematisch Instituut 3584 CD Utrecht

Solutions Final: Inleiding Financiele Wiskunde 2019-2020

(1) Let $\{W(t) : t \geq 0\}$ be a Brownian motion with filtration $\{\mathcal{F}(t) : t \geq 0\}$. Consider the process $\{S(t) : t \geq 0\}$ defined by

$$S(t) = -\int_0^t 2S(u)\,du + \int_0^t e^{-4u}\,dW(u).$$

(a) Show that the process $\{e^{2t}S(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. (1 pt)

(b) Determine the distribution of $S(t)$. (1 pt)

Proof (a) : First observe that

$$d(S(t)) = -2\,S(t)\,dt + e^{-4t}\,dW(t).$$

We apply Itô product rule, we get

$$\begin{aligned} d\Big(e^{2t}S(t)\Big) &= 2e^{2t}S(t)\,dt + e^{2t}\,dS(t) \\ &= 2e^{2t}S(t)\,dt + e^{2t}\Big(-2\,S(t)\,dt + e^{-4t}\,dW(t)\Big) \\ &= e^{-2t}\,dW(t). \end{aligned}$$

Since $S(0) = 0$, we see that $e^{2t}S(t) = \int_0^t e^{-2u}\,dW(u)$ is an Itô-integral and hence the process $\{e^{2t}S(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$.

Proof (b) : From part (a), we have $S(t) = \int_0^t e^{-2(t+u)}\,dW(u)$. Note that $\int_0^t e^{-2(t+u)}\,dW(u)$ is an Itô-integral of a deterministic process, hence it is normally distributed with mean 0 and variance

$$\text{Var}\Big(\int_0^t e^{-2(t+u)}\,dW(u)\Big) = e^{-4t}\int_0^t e^{-4u}\,du = \frac{e^{-4t}(1-e^{-4t})}{4}.$$

(2) Let $\{(W_1(t), W_2(t)) : t \geq 0\}$ be a 2-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the price process $\{S(t) : t \geq 0\}$ given by

$$S(t) = 1 + \int_0^t \alpha S(u)\,dW_1(u) + \int_0^t \beta S(u)\,dW_2(u)$$

where α, β are positive constants.

(a) Show that $\{S^2(t) : t \geq 0\}$ is a 2-dimensional Itô-process. (1pt)

(b) Show that $\mathbb{E}[S^2(t)] = e^{(\alpha^2+\beta^2)t}$, $t \geq 0$. (You are allowed to interchange integrals and expectations but justify why). (1.5 pts)

Proof (a) : From the hypothesis, we have $d(S(t)) = \alpha S(t)dW_1(t) + \beta S(t)dW_2(t)$. Consider the continuously differentiable function $f(x) = x^2$. We have $f_x(x) = 2x, f_{xx}(x) = 2$ and $S^2(t) = f(S(t))$. By Itô-Doeblin,

$$\begin{aligned} d((S^2(t)) &= 2S(t)dS(t) + \frac{1}{2}\cdot 2dS(t)dS(t) \\ &= 2S(t)\Big(\alpha S(t)dW_1(t) + \beta S(t)dW_2(t)\Big) + (\alpha^2+\beta^2)S^2(t)dt \\ &= (\alpha^2+\beta^2)S^2(t)dt + 2\alpha S^2(t)dW_1(t) + 2\beta S^2(t)dW_2(t). \end{aligned}$$

Since $S(0) = 1$, we have $S^2(0) = 1$ and $S^2(t) = 1 + \int_0^t (\alpha^2 + \beta^2) S^2(u)du + \int_0^t 2\alpha S^2(u)dW_1(u) + \int_0^t 2\beta S^2(u)dW_2(u)$. Hence $\{S^2(t) : t \geq 0\}$ is a 2-dimensional Itô-process.

Proof (b) : From part (a),

$$S^2(t) = 1 + \int_0^t (\alpha^2 + \beta^2) S^2(u)du + \int_0^t 2\alpha S^2(u)dW_1(u) + \int_0^t 2\beta S^2(u)dW_2(u).$$

Since Itô-integrals have zero expectation, we see by Fubini's Theorem that

$$\mathbb{E}[S^2(t)] = 1 + \mathbb{E}\Big[\int_0^t (\alpha^2 + \beta^2) S^2(u)du\Big] = 1 + \int_0^t (\alpha^2 + \beta^2)\mathbb{E}[S^2(u)]du.$$

Let $m(t) = \mathbb{E}[S^2(t)]$, then the above equation reads

$$m(t) = 1 + \int_0^t (\alpha^2 + \beta^2) m(u)du,$$

or equivalently $\frac{dm(t)}{dt} = (\alpha^2 + \beta^2)m(t)$ with $m(0) = 1$. This differential equation has solution $m(t) = Ce^{(\alpha^2+\beta^2)t}$. Since $m(0) = 1$, we have $C = 1$ and hence $m(t) = \mathbb{E}[S^2(t)] = e^{(\alpha^2+\beta^2)t}$.

(3) Let T fe finite horizon and let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}(t) : 0 \leq t \leq T\}$, where $\mathcal{F}(T) = \mathcal{F}$. Suppose that the price process $\{S(t) : 0 \leq t \leq T\}$ of a certain stock is given by

$$S(t) = \exp\Big\{\int_0^t (1+u)\, dW(u) + t - \frac{t^3}{6}\Big\}$$

(a) Show that $\{S(t) : 0 \leq t \leq T\}$ is an Itô-process. (1 pt)

(b) Let r be a constant interest rate. Find a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that the discounted process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\widetilde{\mathbb{P}}$. (1.5 pts)

Proof (a): First note that $S(t) = \exp\Big\{\int_0^t (1+u)\, dW(u) + \int_0^t (1 - \frac{u^2}{2})\, du\Big\}$. Let

$$X(t) = \int_0^t (1+u)\, dW(u) + \int_0^t (1 - \frac{u^2}{2})\, du,$$

then $X(0) = 0$ and $dX(t) = (1+t)\, dW(t) + (1 - \frac{t^2}{2})\, dt$. Consider the continuously differentiable function $f(x) = e^x$. We have $f_x(x) = f_{xx}(x) = e^x$ and $S(t) = f(X(t)) = f_x(X(t)) = f_{xx}(X(t))$. By Itô-Doeblin,

$$\begin{aligned} d(S(t)) = d(f(X(t))) &= S(t)\, dX(t) + \frac{1}{2}S(t)\, dX(t)dX(t) \\ &= S(t)\Big((1+t)dW(t) + (1 - \frac{1}{2}t^2)dt\Big) + \frac{1}{2}S(t)(1+t)^2 dt \\ &= S(t)(\frac{3}{2} + t)dt + (1+t)S(t)dW(t). \end{aligned}$$

Hence, $S(t) = 1 + \int_0^t (\frac{3}{2} + u)S(u)\, du + \int_0^t (1+u)S(u)\, dW(u)$, and therefore $\{S(t) : 0 \leq t \leq T\}$ is an Itô-process.

Proof (b): We consider the adapted process $\{\theta(t) : 0 \leq t \leq T\}$ given by $\theta(t) = \frac{\frac{3}{2}+t-r}{1+t}$, and the random variable

$$Z = \exp\Big\{-\int_0^T \theta(u)\, dW(u) - \frac{1}{2}\int_0^T \theta^2(u)\, du.\Big\}$$

Notice that $\theta(t)$ is bounded on the interval $[0, T]$, hence $\mathbb{E}\Big[\int_0^T \theta^2(u)Z^2(u)\, du\Big] < \infty$. Consider the probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ defined by $\widetilde{\mathbb{P}}(A) = \int_A Z\, d\mathbb{P}$, and the process $\{\widetilde{W}(t) : 0 \leq$

$t \le T\}$ with $\widetilde{W}(t) = \int_0^t \theta(u)\,du + W(t)$. By Girsanov's Theorem, the process $\{\widetilde{W}(t) : 0 \le t \le T\}$ is a Brownian motion under $\widetilde{\mathbb{P}}$, and by Itô product rule we have,

$$\begin{aligned}
d(e^{-rt}S(t)) &= e^{-rt}\,dS(t) - re^{-rt}S(t)\,dt \\
&= e^{-rt}[(\frac{3}{2}+t)S(t)\,dt + (1+t)S(t)\,dW(t)] - re^{-et}S(t)\,dt \\
&= e^{-rt}(\frac{3}{2}+t-r)S(t)\,dt + e^{-rt}(1+t)S(t)\,dW(t) \\
&= e^{-rt}(1+t)\theta(t)S(t)\,dt + e^{-rt}(1+t)S(t)\,dW(t) \\
&= e^{-rt}(1+t)S(t)\Big(\theta(t)\,dt + dW(t)\Big) \\
&= e^{-rt}(1+t)S(t)d\widetilde{W}(t).
\end{aligned}$$

This shows that the process $\{e^{-rt}S(t) : 0 \le t \le T\}$ is an Itô process under $\widetilde{\mathbb{P}}$ and hence a martingale under $\widetilde{\mathbb{P}}$.

(4) Let T be a finite time (expiration date) and let $\{(W_1(t), W_2(t) : 0 \le t \le T)$ be a two-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}(t) : 0 \le t \le T\}$, where $\mathcal{F} = \mathcal{F}(T)$. Consider two price processes

$$\begin{aligned}
dS_1(t) &= S_1(t)\,dt + 0.3S_1(t)\,dW_1(t) + 0.3S_1(t)\,dW_2(t) \\
dS_2(t) &= 2S_2(t)\,dt + 0.1S_2(t)\,dW_1(t).
\end{aligned}$$

We assume $S_1(0), S_2(0) > 0$.

(a) Assume that the interest rate is a constant, i.e. $R(t) = r$ for $t > 0$. Find the unique risk-neutral probability $\widetilde{\mathbb{P}}$, i.e. the probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ under which the discounted price processes $\{e^{-rt}S_i(t) : 0 \le t \le T\}$ are martingales, $i = 1, 2$. (1.5 pts)

(b) Consider a financial derivative with payoff at time T given by $V(T) = \dfrac{1}{T}\displaystyle\int_0^T S_2(t)\,dt$. Show that the *fair* price at time 0 of this derivative is given by $V(0) = \frac{S_2(0)}{rT}(1 - e^{-rT})$. (1.5 pts)

Proof (a): We will show that the market price equations have a unique solution. Using the notation of the book we have, $\alpha_1 = 1, \sigma_{11} = 0.3, \sigma_{12} = 0.3, \alpha_2 = 2, \sigma_{21} = 0.1, \sigma_{22} = 0$ and the market price equations are given by

$$\begin{aligned}
1 - r &= 0.3\theta_1(t) + 0.3\theta_2(t) \\
2 - r &= 0.1\theta_1(t).
\end{aligned}$$

This system has a unique solution given by $\theta_1(t) = 10(2-r)$ and $\theta_2(t) = \frac{10(2r-5)}{3}$. Setting

$$\begin{aligned}
Z &= \exp\Big\{ -\int_0^T \Big(\theta_1(t)\,dW_1(t) + \theta_2(t)\,dW_2(t)\Big) - \frac{1}{2}\int_0^T \Big(\theta_1^2(t) + \theta_2^2(t)\Big)\,dt\Big\} \\
&= \exp\Big\{ -10(2-r)W_1(T) - \frac{10(2r-5)}{3}W_2(T) - \Big(50(2-r)^2 + \frac{50(2r-5)^2}{9}\Big)T\Big\},
\end{aligned}$$

the desired risk-neutral probability measure is given by $\widetilde{\mathbb{P}}(A) = \int_A Z\,d\mathbb{P}$. To check this, we set

$$\widetilde{W}_1(t) = \int_0^t \theta_1(u)du + W_1(t) = 10(2-r)t + W_1(t),$$

and

$$\widetilde{W}_2(t) = \int_0^t \theta_2(u)du + W_2(t) = \frac{10(2r-5)}{3}t + W_2(t).$$

By the 2-dimensional Girsanov Theorem the process $\{(\widetilde{W}_1(t), \widetilde{W}_1(t)) : 0 \le t \le T\}$ is a 2-dimensional Brownian motion. Applying Itô product rule on $e^{-rt}S_1(t), e^{-rt}S_2(t)$ and rewriting in terms of $\widetilde{W}_1(t), \widetilde{W}_1(t)$, we get

$$\begin{aligned}
d(e^{-rt}S_1(t)) &= e^{-rt}S_1(t)(0.3d\widetilde{W}_1(t) + 0.3d\widetilde{W}_2(t)) \\
d(e^{-rt}S_2(t)) &= e^{-rt}S_2(t)0.1d\widetilde{W}_1(t),
\end{aligned}$$

which shows that the discounted price processes are martingales under $\widetilde{\mathbb{P}}$.

Proof (b): From part (a), we see that the Market price equations have a unique solution and hence the risk neutral measure $\widetilde{\mathbb{P}}$ is unique. By the Second Fundamental Theorem of Asset Pricing, the market is complete and every derivative security can be hedged. Hence, there exists a (self-financing) portfolio $\{X(t) : 0 \leq t \leq T\}$ such that $X(t) = V(t)$ for all $0 \leq t \leq T$. Since $\{e^{-rt}X(t) : 0 \leq t \leq T\}$, $\{e^{-rt}S_1(t) : 0 \leq t \leq T\}$ and $\{e^{-rt}S_2(t) : 0 \leq t \leq T\}$ are all martingales under $\widetilde{\mathbb{P}}$, we see that the price at time zero is given by

$$\begin{aligned}
X(0) = V(0) &= \widetilde{\mathbb{E}}[e^{-rT}V(T)] \\
&= \widetilde{\mathbb{E}}\Big[\frac{e^{-rT}}{T}\int_0^T S_2(t)\,dt\Big] \\
&= \frac{e^{-rT}}{T}\int_0^T e^{rt}\widetilde{\mathbb{E}}[e^{-rt}S_2(t)]\,dt \\
&= \frac{e^{-rT}}{T}\int_0^T e^{rt}S_2(0)\,dt \\
&= \frac{e^{-rT}S_2(0)}{rT}(e^{rT}-1) \\
&= \frac{S_2(0)}{rT}(1-e^{-rT}),
\end{aligned}$$

the third equality follows from Fubini's (in fact Tonelli's) Theorem since $S_2(t) = S_2(0)\exp\Big\{0.1W_1(t) + (2 - \frac{1}{2}(0.1)^2)t\Big\} > 0$ and the fourth equality follows from the fact that $\{e^{-rt}S_2(t) : 0 \leq t \leq T\}$ is a martingales under $\widetilde{\mathbb{P}}$.