Universiteit Utrecht Boedapestlaan 6
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Retake Exam: Inleiding Financiele Wiskunde 2019-2020

(1) Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Let $c(t)$ be a deterministic function with $c(0) = 0$. Let r be a given interest rate and consider the price process $\{S(t) : 0 \leq t \leq T\}$ given by $S(t) = e^{W(t)+c(t)}$.

(a) Find an expression for $c(t)$ under which the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale with respect to the probability measure $\mathbb{P}$. (1.5 pts)

(b) Consider the expression obtained in part (a), i.e. assume that the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under the probability measure $\mathbb{P}$. Consider a financial derivative with payoff at time T given by $V(T) = \mathbb{I}_{\{S(T)>K\}}$ (i.e. $V(T)$ has value 1 if $S(T) > K$ and 0 otherwise), here K is some given positive constant. Find the *fair* price of this option at time 0. (1.5 pts)

(2) Let $\{W_1(t), W_2(t)) : t \geq 0\}$ be a two dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the two processes $\{Z(t) : t \geq 0\}$ and $\{B(t) : t \geq 0\}$ defined by

$$Z(t) = 1 + e^{-W_1(t)} \int_0^t e^{W_1(u)} dW_2(u)$$

and

$$B(t) = \int_0^t \frac{1}{\sqrt{1 + Z^2(u)}}\, dW_1(u) - \int_0^t \frac{Z(u)}{\sqrt{1 + Z^2(u)}}\, dW_2(u).$$

(a) Use Lévy's characterization to prove that the process $\{B(t) : t \geq 0\}$ is a one dimensional Brownian motion. (1 pt)

(b) Prove that the process $\{Z(t) : t \geq 0\}$ can be written as

$$Z(t) = 1 + W_2(t) - \int_0^t Z(u)\, dW_1(u) + \frac{1}{2} \int_0^t Z(u)\, ds.$$

(1.5 pts)

(c) Prove that $\mathbb{E}[Z(t)] = e^{\frac{1}{2}t}$, for $t \geq 0$. (1 pt)

(3) Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$. Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$ with

$$S(t) = S(0) \exp\Big\{ \int_0^t e^{-u}\, dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2u})\, du \Big\}.$$

(a) Let $X(t) = \int_0^t e^{-u}\, dW(u) + \int_0^t (1 - \frac{1}{2}e^{-2u})\, du$. Determine the distribution of $X(t)$. (1 pt)

(b) Prove that $\{S(t) : t \geq 0\}$ is an Itô process. (1 pt)

(c) Let r be a constant interest rate. Find the risk-neutral measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e. $\widetilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0$, $A \in \mathcal{F}$) such that the discounted price process $\{e^{-rt}S(t) : 0 \leq t \leq T\}$ is a martingale under $\widetilde{\mathbb{P}}$. (1.5 pts)