Utrecht University
Mathematics
Stochastic processes
Fall 2011

Test, November 6, 2012

JUSTIFY YOUR ANSWERS

Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of five questions plus one bonus problem.
- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. Let X_i, $1 = 1, \ldots, n$ be independent normal random variables with respective means μ_i and variances σ_i^2. Consider its mean

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

(a) (0.5 pts.) Prove that $\overline{X}$ is also normally distributed.

(b) (0.5 pts.) Determine the mean and variance of $\overline{X}$.

✓ **Exercise 2.** (1 pt.) Consider a branching process with offspring number with mean μ and variance σ. That means, a sequence of random variables $(X_n)_{n\geq 0}$ with $X_0 = 1$ and

$$X_n = \sum_{n=1}^{X_{n-1}} Z_i \qquad n \geq 1$$

where Z_n are iid random variables (offspring distribution) independent of the (X_n) with mean μ. Show that $E(X_n) = \mu^n$. [*Hint:* Start by showing that $E(X_n) = \mu\, E(X_{n-1})$.]

✓ **Exercise 3.**

(a) (0.8 pts.) Show that

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}^n = \begin{pmatrix} 1/2 + a^n/2 & 1/2 - a^n/2 \\ 1/2 - a^n/2 & 1/2 + a^n/2 \end{pmatrix}$$

for $n \geq 1$. Determine a.

(b) A communication system transmits the digits 0 and 1. Each digit must pass through n stages, each of which independently transmits the digit correctly with probability p.

-i- (0.8 pts.) Find the probability that the final digit, X_n, is correct.

-ii- (0.8 pts.) Find the probability that *all* the first n stages transmit correctly.

(Turn over, please)

Exercise 4. Consider a three-state Markov process $(X_n)_{n\geq 0}$ with two absorbing states. That is, a process with a three-symbol alphabet (=state space), say $\{0,1,2\}$, and transition matrix

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 \\ a & b & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c > 0$ and $a+b+c = 1$.

(a) (0.8 pts.) Show that $\mathbb{P}^n_{11} = b^n$.

(b) (0.8 pts.) Show that the state "1" is transient.

(c) (0.8 pts.) Let $T = \inf\{n > 0 : X_n = 0 \text{ or } X_n = 2\}$ be the time it takes the process to be absorbed in one of the absorbing states. Compute $E(T \mid X_0 = 1)$. [*Hint:* you may want to use that for a discrete random variable Z, $E[Z] = \sum_{k\geq 0} P(Z > k)$.]

(d) (0.8 pts.) Let $T_0 = \inf\{n > 0 : X_n = 0\}$ and $T_2 = \inf\{n > 0 : X_n = 2\}$ be the absorption times at each of the absorbing states. Compute $P(T_0 < T_2 \mid X_0 = 1)$.

(e) (0.8 pts.) Compute *all* the invariant measures of the process.

Exercise 5. At a certain beach resort a bad day is equally likely to be followed by a good or a bad day, while a good day is five times more likely to be followed by a good day than by a bad day. The number of interventions by lifesavers is Poisson distributed with mean 4 in good days and mean 1 in bad days. Find, in the long run,

(a) (0.8 pts.) The probability of the lifesavers not having any intervention in a given day.

(b) (0.8 pts.) The average number of interventions per day.

[Take $e^{-4} \sim 0.02$ and $e^{-1} \sim 0.4$.]

Bonus problem

Bonus Consider a homogeneous (or shift-invariant) Markov chain $(X_n)_{n\in\mathbb{N}}$ $(X_n)_{n\in\mathbb{N}}$ with *finite* state space S. Let us recall that the *hitting time* of a state y is

$$T_y = \min\{n \geq 1 : X_n = y\}.$$

(a) If $\ell \leq n \in \mathbb{N}$, $x, y \in S$, prove the following

-i- (0.5 pts.)

$$P\big(X_n = y, T_y = \ell \mid X_0 = x\big) = P^{n-\ell}_{yy}\, P\big(T_y = \ell \mid X_0 = x\big).$$

-ii- (0.5 pts.)

$$P^n_{xy} = \sum_{\ell=1}^{n} P^{n-\ell}_{yy}\, P\big(T_y = \ell \mid X_0 = x\big).$$

(b) Conclude the following:

-i- (0.5 pts.) If every state is transient, then for every $x, y \in S$.

$$\sum_{n\geq 0} P^n_{xy} < \infty.$$

-ii- (0.5 pts.) The previous result leads to a contradiction with the stochasticity property of the matrix $\mathbb{P}$. Hence not all states can be transient.