JUSTIFY YOUR ANSWERS
Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of five questions plus one bonus problem.

- The score is computed by adding all the credits up to a maximum of 10

Exercise 1. Let X_i, $1 = 1, \ldots, n$ be independent normal random variables with respective means μ_i and variances σ_i^2. Consider its mean

$$\overline{X} \;=\; \frac{1}{n} \sum_{i=1}^{n} X_i$$

(a) (0.5 pts.) Prove that $\overline{X}$ is also normally distributed.

(b) (0.5 pts.) Determine the mean and variance of $\overline{X}$.

Answers: *The moment-generating function $\Phi_{\overline{X}}(t)$ of $\overline{X}$ factorizes, due to the independence of the X_i, in the following way:*

$$\Phi_{\overline{X}}(t) \;=\; E\big(\mathrm{e}^{t\overline{X}}\big) \;=\; \prod_{i=1}^{n} E\big(\mathrm{e}^{tX_i/n}\big) \;=\; \prod_{i=1}^{n} \Phi_{X_i}(t/n)$$

where Φ_{X_i} is the moment-generator function of the variable X_i. As each X_i is normal,

$$\Phi_{X_i}(s) \;=\; \exp\Big[\mu_i s + \frac{\sigma_i^2 s^2}{2}\Big]\,,$$

hence

$$\Phi_{\overline{X}}(t) \;=\; \exp\Big[t\Big(\frac{1}{n}\sum_{i=1}^{n}\mu_i\Big) + \frac{t^2}{2}\Big(\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2\Big)\Big]\,.$$

This is the moment-generating function of a normal variable with mean $\frac{1}{n}\sum_{i=1}^{n}\mu_i$ and variance $\frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$. This identifies $\overline{X}$ as a variable with such a law.

Exercise 2. (1 pt.) Consider a branching process with offspring number with mean μ and variance σ. That means, a sequence of random variables $(X_n)_{n\geq 0}$ with $X_0 = 1$ and

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i \qquad n \geq 1$$

where Z_n are iid random variables (offspring distribution) independent of the (X_n) with mean μ Show that $E(X_n) = \mu^n$. [*Hint:* Start by showing that $E(X_n) = \mu\, E(X_{n-1})$.]

Answer: *Start with*

$$E(X_n) \;=\; E\big(E(X_n \mid X_{n-1})\big)\,.$$

Now

$$E(X_n \mid X_{n-1} = x_{n-1}) \;=\; E\Big(\sum_{n\geq 1}^{x_{n-1}} Z_i \;\Big|\; X_{n-1} = x_{n-1}\Big)$$

$$=\; \sum_{n\geq 1}^{x_{n-1}} E(Z_i \mid X_{n-1} = x_{n-1})$$

$$=\; \sum_{n\geq 1}^{x_{n-1}} E(Z_i) \qquad \text{(independence of } Z_i \text{ and } X_{n-1})$$

$$=\; x_{n-1}\,\mu \,.$$

Hence $E(X_n \mid X_{n-1}) = \mu\, X_{n-1}$ and

$$E(X_n) \;=\; E\big(\mu\, X_{n-1}\big) \;=\; \mu\, E(X_{n-1})\,.$$

By induction in n we get the proposed result.

Exercise 3.

(a) (0.8 pts.) Show that

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}^n = \begin{pmatrix} 1/2 + a^n/2 & 1/2 - a^n/2 \\ 1/2 - a^n/2 & 1/2 + a^n/2 \end{pmatrix}$$

for $n \geq 1$. Determine a.

Answer: *This is an easy proof by induction. Comparing for the case $n = 1$ we obtain $a = 2p - 1$.*

(b) A communication system transmits the digits 0 and 1. Each digit must pass through n stages, each of which independently transmits the digit correctly with probability p.

-i- (0.8 pts.) Find the probability that the final digit, X_n, is correct.
Answer: $P_{00}^n = P_{11}^n = 1/2 - (2p-1)^n/2$.

-ii- (0.8 pts.) Find the probability that *all* the first n stages transmit correctly.
Answer: *By independence the probability is equal to p^n.*

Exercise 4. Consider a three-state Markov process $(X_n)_{n\geq 0}$ with two absorbing states. That is, a process with a three-symbol alphabet (=state space), say $\{0, 1, 2\}$, and transition matrix

$$\mathbb{P} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ a & b & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c > 0$ and $a + b + c = 1$.

(a) (0.8 pts.) Show that $\mathbb{P}_{11}^n = b^n$.
Answer: *As $P_{x\,1} = 0$ for every $x \neq 1$,*

$$P_{11}^n \;=\; \sum_{x=0}^{2} P_{1\,x}^{n-1} P_{x\,1} \;=\; P_{11}^{n-1} P_{11} \;=\; P_{11}^{n-1}\, b\,.$$

The result follows by induction.

(b) (0.8 pts.) Show that the state "1" is transient.

Answer:

$$\sum_{n \geq 0} P_{11}^n = \sum_{n \geq 0} b^n = \frac{1}{1-b} < \infty.$$

(c) (0.8 pts.) Let $T = \inf\{n > 0 : X_n = 0 \text{ or } X_n = 2\}$ be the time it takes the process to be absorbed in one of the absorbing states. Compute $E(T \mid X_0 = 1)$. [*Hint:* you may want to use that for a discrete random variable Z, $E[Z] = \sum_{k \geq 0} P(Z > k)$.]

Answer:

$$E(T \mid X_0 = 1) =$$
$$\sum_{k \geq 0} P(T > k \mid X_0 = 1) = \sum_{k \geq 0} P\big(X_n = 1, n = 1, \dots, k \mid X_0 = 1\big) = \sum_{k \geq 0} b^k = \frac{1}{1-b}.$$

(d) (0.8 pts.) Let $T_0 = \inf\{n > 0 : X_n = 0\}$ and $T_2 = \inf\{n > 0 : X_n = 2\}$ be the absorption times at each of the absorbing states. Compute $P(T_0 < T_2 \mid X_0 = 1)$.

Answer:

$$P(T_0 < T_2 \mid X_0 = 1)$$
$$= \sum_{k \geq 0} P\big(X_{k+1} = 0, X_n = 1, n = 1, \dots, k \mid X_0 = k\big) = \sum_{k \geq 0} P\big(X_{k+1} = 0 \mid X_k = 1\big) P_{11}^k$$
$$= \sum_{k \geq 0} a\, b^k = \frac{a}{1-b}.$$

(e) (0.8 pts.) Compute *all* the invariant measures of the process.

Answer: *Let $\pi = (\pi_0, \pi_1, \pi_2)$ be the invariant measure. The conditions $\sum_x \Pi_x P_{xy} = \Pi_x$ plus the normalisation condition $\Pi_0 + \Pi_1 + \Pi_2 = 1$ become:*

$$\Pi_0 + a\,\Pi_1 = \Pi_0$$
$$b\,\Pi_1 = \Pi_1$$
$$c\,\Pi_1 + \Pi_2 = \Pi_2$$
$$\Pi_0 + \Pi_1 + \Pi_2 = 1.$$

All their solutions are of the form $\Pi_1 = 0$, $\Pi_1 + \Pi_2 = 1$. That is, the invariant measures Π take the form

$$\Pi = (\lambda, 0, 1 - \lambda) = \lambda\,(1, 0, 0) + (1 - \lambda)(0, 0, 1) \quad \text{for } 0 \leq \lambda \leq 1,$$

That is, the invariant measures are convex combinations of the measure concentrated in the state "0" and the measure concentrated in the state "2".

Exercise 5. At a certain beach resort a bad day is equally likely to be followed by a good or a bad day, while a good day is five times more likely to be followed by a good day than by a bad day. The number of interventions by lifesavers is Poisson distributed with mean 4 in good days and mean 1 in bad days. Find, in the long run,

(a) (0.8 pts.) The probability of the lifesavers not having any intervention in a given day.

(b) (0.8 pts.) The average number of interventions per day.

[Take $e^{-4} \sim 0.02$ and $e^{-1} \sim 0.4$.]

Answers: *Associating "good days" $\to$ 1 and "bad days" $\to$ 2, the weather pattern is a Markov process with transition matrix*

$$\begin{pmatrix} 5/6 & 1/6 \\ 1/2 & 1/2 \end{pmatrix}.$$

The proportion of good and bad days is determined, in the long run, by the invariant measure Π of this chain. This measure satisfies:

$$\left.\begin{array}{r} \frac{5}{6}\Pi_1 + \frac{1}{2}\Pi_2 = \Pi_1 \\ \Pi_1 + \Pi_2 = 1 \end{array}\right\} \implies \Pi = \left(\frac{3}{4}, \frac{1}{4}\right).$$

(a) *Let S be the number of interventions per day.*

$$\begin{aligned} P(S=0) &= P(S=0 \mid good\ day)\,P(good\ day) + P(S=0 \mid bad\ day)\,P(bad\ day) \\ &= e^{-4}\frac{3}{4} + e^{-1}\frac{1}{4} \sim 0.11 \end{aligned}$$

(b)

$$\begin{aligned} E(S) &= E(S \mid good\ day)\,P(good\ day) + E(S \mid bad\ day)\,P(bad\ day) \\ &= 4 \cdot \frac{3}{4} + 1 \cdot \frac{1}{4} = \frac{13}{4} = 3.25 . \end{aligned}$$

Bonus problem

Bonus Consider a homogeneous (or shift-invariant) Markov chain $(X_n)_{n \in \mathbb{N}}$ $(X_n)_{n \in \mathbb{N}}$ with *finite* state space S. Let us recall that the *hitting time* of a state y is

$$T_y = \min\{n \geq 1 : X_n = y\}.$$

(a) If $\ell \leq n \in \mathbb{N}$, $x, y \in S$, prove the following

-i- (0.5 pts.)
$$P\big(X_n = y, T_y = \ell \mid X_0 = x\big) = P_{yy}^{n-\ell}\, P\big(T_y = \ell \mid X_0 = x\big).$$

Answer: *Decomposing in terms of trajectories,*

$$\begin{aligned} &P\big(X_n = y, T_y = \ell \mid X_0 = x\big) \\ &= \sum_{x_1,\dots,x_{\ell-1} \neq y} P\big(X_n = y, X_\ell = y, X_{\ell-1} = x_{\ell-1}, \cdots, X_1 = x_1 \mid X_0 = x\big) \\ &= \sum_{x_1,\dots,x_{\ell-1} \neq y} P_{yy}^{n-\ell} P\big(X_\ell = y, X_{\ell-1} = x_{\ell-1}, \cdots, X_1 = x_1 \mid X_0 = x\big) \\ &= \sum_{x_1,\dots,x_{\ell-1} \neq y} P_{yy}^{n-\ell} P\big(T_y = \ell \mid X_0 = x\big) \end{aligned}$$

-ii- (0.5 pts.)
$$P_{xy}^n = \sum_{\ell=1}^{n} P_{yy}^{n-\ell}\, P\big(T_y = \ell \mid X_0 = x\big).$$

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Answer: *As*

$$\{X_n = y\} = \bigcup_{\ell=1}^{n} \{X_n = y, T_y = \ell\},$$

the union being disjoint, we conclude that

$$\sum_{\ell=1}^{n} P(X_n = y, T_y = \ell \mid X_0 = x) = P(X_n = y \mid X_0 = x) = P_{xy}^n.$$

The result follows, hence, by summing both sides of -i- with respect to ℓ.

(b) Conclude the following:

-i- (0.5 pts.) If every state is transient, then for every $x, y \in S$.

$$\sum_{n \geq 0} P_{xy}^n < \infty.$$

Answer: *By -ii- above,*

$$\sum_{n \geq 0} P_{xy}^n = \sum_{n \geq 0} \sum_{\ell=1}^{n} P_{yy}^{n-\ell} P(T_y = \ell \mid X_0 = x).$$

Hence, interchanging the order of summation,

$$\begin{aligned}
\sum_{n} P_{xy}^n &= \sum_{\ell \geq 1} \sum_{n \geq \ell} P_{yy}^{n-\ell} P(T_y = \ell \mid X_0 = x) \\
&= \sum_{\ell \geq 1} P(T_y = \ell \mid X_0 = x) \sum_{m \geq 0} P_{yy}^m \\
&= P(T_y < \infty \mid X_0 = x) \sum_{m \geq 0} P_{yy}^m.
\end{aligned}$$

If y is transient, the last sum is finite.

-ii- (0.5 pts.) The previous result leads to a contradiction with the stochasticity property of the matrix $\mathbb{P}$. Hence not all states can be transient.

Answer: *Summing over y the inequality in (b)-i- we get*

$$\sum_{y} \sum_{n \geq 0} P_{xy}^n < \infty \tag{1}$$

(recall that S is finite). However, by stochasticity $\sum_{y} P_{xy}^n = 1$ for every $n \geq 0$. Hence,

$$\sum_{y} \sum_{n \geq 0} P_{xy}^n = \sum_{n \geq 0} \sum_{y} P_{xy}^n = \sum_{n \geq 0} 1 = \infty,$$

in contradiction with (1).