Universiteit Utrecht
Blok 1, 2017/18
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Wiskundige Technieken 3

NS-220B

Final Exam

Name:

Student number:

Date: Thursday, November 9, 2017

Time: 9:00 - 12:00 (3 hours)

Room: OLYMPOS, HAL1

Instructions:

- Write your *name, student number, and problem number* on every page you hand in.
- Use a *separate* sheet for each problem.
- The use of textbooks, notes, calculators, cell phones, etc. is *not* allowed.
- Make sure that your answers are *readable* and *understandable.*
- Problems marked with * are bonus questions.

Total points: 48 (including bonus points)

Score:

1	2	3	4	5	Σ

Grade:

Problem 1.

a) Determine all eigenvalues and eigenvectors of the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix}.$$

3p

b) Show that the matrix A is diagonalizable. 2p

c) Use the results from a) and b) to find the solution $F : \mathbb{R} \to \mathbb{C}^2$ to

$$\frac{d}{dt} F = AF$$

that satisfies the initial condition $F(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$. 2p

Problem 2.

Consider the differential equation

$$f'' + (2 - 4x^2) f = 0 \tag{1}$$

for $f : \mathbb{R} \to \mathbb{R}$.

a) Assume that the power series function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad x \in \mathbb{R},$$

with $a_n \in \mathbb{R}$ solves (1). Plug this ansatz into (1) and derive a recurrence relation for the coefficients a_n.
Hint: Treat the zeroth and first order terms separately. Notice that solving the recurrence relation is not required. 3p

Take $a_0 = 1$ and $a_1 = 0$.

b) Argue that $a_n = 0$ for all odd positive integers n. 2p

Regarding the even positive integers, you may use without proof that

$$a_n = \frac{(-1)^k}{k!} \qquad \text{for } n = 2k \text{ with } k \in \mathbb{N}.$$

c) Use a convergence test of your choice to prove that the power series $\sum_{n \geq 0} a_n x^n$ converges for all $x \in \mathbb{R}$. 3p

d) Show that $f(x) = e^{-x^2}$ for all $x \in \mathbb{R}$ and verify that this function is actually a solution to (1). 3p

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Problem 3.

Let $f : \mathbb{R} \to \mathbb{R}$ be the 2-periodic function defined by

$$f(x) = \begin{cases} 0 & \text{for } x \in (-1, 0], \\ x & \text{for } x \in (0, 1), \\ \frac{1}{2} & \text{for } x = 1, \end{cases} \qquad x \in (-1, 1].$$

a) Visualize the graph of the function f by drawing at least two periods. 1p

b) Determine the Fourier coefficients $\hat{f}_k$ for all $k \in \mathbb{Z}$. 3p

c) Argue why f can be expressed as a converging Fourier sine and cosine series, i.e.

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x) \qquad \text{for all } x \in \mathbb{R}$$

with $a_k, b_k \in \mathbb{R}$, and determine all the coefficients a_k and b_k. 3p

d)* Use the result from c) to find the value of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots.$$

2p

Problem 4.

Consider the transport equation

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} u = 0 \qquad \text{in } \mathbb{R} \times (0, \infty), \tag{2}$$

along with the initial conditions

$$u(x, 0) = g(x) \qquad \text{for } x \in \mathbb{R}, \tag{3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a given function.

a) Let $g(x) = e^{2x}$ for $x \in \mathbb{R}$. Use the separation of variables method to find a function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ that satisfies (2) and (3). 4p

b) Show that for all continuously differentiable functions g,

$$u(x, t) = g(x - t) \qquad \text{with } x \in \mathbb{R} \text{ and } t \in [0, \infty), \tag{4}$$

solves the initial value problem (2)–(3). 2p

c)* Assuming that $g(x) = \sin(x)$ for $x \in \mathbb{R}$, is it possible to derive the solution (4) in b) via a separation of variables approach? 2p

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Problem 5.

We consider the linear second order differential equation

$$v'' + 4v' + 3v = f \tag{5}$$

for $v : \mathbb{R} \to \mathbb{R}$ with

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(t) = \begin{cases} 2 & \text{if } 0 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Solve the differential equation by performing the following steps.

a) Apply the Fourier transformation $\mathcal{F}$ to (5) and solve the resulting equation for $\mathcal{F}v = \hat{v}$. 3p

b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} \frac{1}{2}(e^{-t} - e^{-3t}) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Find the Fourier transform $\mathcal{F}g = \hat{g}$ of g. 2p

c) Calculate the convolution product $(f * g)(x)$ for all $x \in \mathbb{R}$.
Hint: Treat the cases $x \leq 0$, $x \in (0, 1)$ and $x \geq 1$ separately. 3p

d) Use the results from a), b) and c) to obtain a particular solution to (5). 3p

e) Determine the general solution to the differential equation (5). 2p