Sketch of suggested solutions

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Problem 1.

a) The characteristic polynomial is $\det(A - \lambda\mathbb{I}) = \lambda^2 + 4$. The eigenvalues are the roots of the characteristic polynomial, hence $\lambda_{1/2} = \pm 2i$.

A non-zero vector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$ is an eigenvector associated with $\lambda_1 = 2i$, if it satisfies $Av = \lambda_1 v$, or equivalently solves the following system of linear equations:

$$\begin{aligned} (1-2i)v_1 + v_2 &= 0, \\ -5v_1 - (1+2i)v_2 &= 0. \end{aligned}$$

Hence,

$$v = c_1 \begin{pmatrix} -1 \\ 1-2i \end{pmatrix}$$

with $c_1 \in \mathbb{C} \setminus \{0\}$. Similarly, one finds that $c_2 \begin{pmatrix} -1 \\ 1+2i \end{pmatrix}$ with $c_2 \in \mathbb{C} \setminus \{0\}$ are the eigenvectors corresponding to $\lambda_2 = -2i$.

b) The matrix A is diagonalizable, since the matrix

$$S = \begin{pmatrix} -1 & -1 \\ 1-2i & 1+2i \end{pmatrix},$$

whose columns consist of eigenvectors associated with λ_1 and λ_2, respectively, is invertible. Indeed, $\det S = -4i \neq 0$. One can show that then $A = SDS^{-1}$ with $D = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$.

c) The general solution to $\frac{d}{dt}F = AF$ is given by

$$F(t) = c_1 e^{2it} \begin{pmatrix} -1 \\ 1-2i \end{pmatrix} + c_2 e^{-2it} \begin{pmatrix} -1 \\ 1+2i \end{pmatrix}$$

for $t \in \mathbb{R}$ with $c_1, c_2 \in \mathbb{C}$. In order to find the solution that satisfies $F(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ we choose the constants c_1 and c_2 such that

$$F(0) = \begin{pmatrix} -c_1 - c_2 \\ c_1(1-2i) + c_2(1+2i) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Hence, $c_2 = -c_1$ and $c_1 = -\frac{1}{2i}$.

Problem 2.

a) Assuming that the convergence radius of the power series function is ∞, we obtain for $x \in \mathbb{R}$ that

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n,$$

and

$$x^2 f(x) = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n.$$

Plugging this into (1) then yields

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 2a_n x^n - \sum_{n=2}^{\infty} 4a_{n-2} x^n = 0.$$

By the identity principle, we conclude that

$$2a_2 + 2a_0 = 0, \qquad 6a_3 + 2a_1 = 0,$$

and

$$(n+2)(n+1)a_{n+2} + 2a_n - 4a_{n-2} = 0 \qquad \text{for } n \geq 2.$$

The latter can be rephrased as

$$a_{n+2} = \frac{4a_{n-2} - 2a_n}{(n+2)(n+1)} \qquad \text{for } n \geq 2,$$

or by an index shift as

$$a_n = \frac{4a_{n-4} - 2a_{n-2}}{n(n-1)} \qquad \text{for } n \geq 4.$$

This is the desired recurrence relation.

b) If $a_0 = 1$ and $a_1 = 0$, we see that $a_2 = -1$ and $a_3 = 0$. Since a_n is given as a linear combination of a_{n-4} and a_{n-2} for every integer $n \geq 4$, iterating this procedure gives that every a_n with n an odd positive integer is a linear combination of a_1 and a_3. Since $a_1 = a_3 = 0$, it follows that $a_n = 0$ for all odd positive integers n.

c) Let us remark that in view of b), $\sum_{n\geq 0} a_n x^n$ converges if and only if $\sum_{k\geq 0} a_{2k} x^{2k}$ converges. Here we choose the ratio test. Alternatively, one can also argue with the comparison test.

Since for every $x \in \mathbb{R}$,

$$\lim_{k\to\infty} \left| \frac{a_{2(k+1)} x^{2(k+1)}}{a_{2k} x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{x^{2k+2} k!}{(k+1)! x^{2k}} \right| = \lim_{k\to\infty} \frac{x^2}{k+1} = 0 < 1,$$

the ratio test gives the convergence of the power series $\sum_{k\geq 0} a_{2k} x^{2k}$ for all $x \in \mathbb{R}$.

d) In view of b) and c),

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_{2k} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}.$$

The right-hand side is the Taylor series expansion of $x \mapsto e^{-x^2}$, hence $f(x) = e^{-x^2}$ for $x \in \mathbb{R}$ as claimed. Since

$$\frac{d}{dx} e^{-x^2} = -2x e^{-x^2} \quad \text{and} \quad \frac{d^2}{dx^2} e^{-x^2} = (4x^2 - 2) e^{-x^2},$$

it is immediate to see that $f(x) = e^{-x^2}$ is a solution to (1).

Problem 3.

a) The visualization of f is left to the reader. We observe that f is piecewise continuously differentiable with jumps in all odd integers, i.e. in $x = 2k + 1$ with $k \in \mathbb{Z}$.

b) For $k = 0$ we have that

$$\hat{f}_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\int_{0}^{1} x\,dx = \frac{1}{4}.$$

In the case $k \neq 0$, we use integration by parts to obtain

$$\begin{aligned}\hat{f}_k &= \frac{1}{2}\int_0^1 xe^{-ik\pi x}\,dx = \frac{1}{2ik\pi}\int_0^1 e^{-ik\pi x}\,dx - \frac{1}{2ik\pi}e^{-ik\pi} \\ &= \frac{1}{2k^2\pi^2}(e^{-ik\pi} - e^0) - \frac{1}{2i\pi k}e^{-ik\pi} = \begin{cases} -\frac{1}{2ik\pi} & \text{if } k \text{ even}, \\ \frac{1}{2ik\pi} - \frac{1}{k^2\pi^2} & \text{if } k \text{ odd}. \end{cases}\end{aligned}$$

c) Since f is piecewise continuously differentiable, the Fourier inversion formula tells us that for $x \in \mathbb{R}$,

$$\frac{1}{2}(f(x^-) + f(x^+)) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{-ik\pi x},$$

where the series on the right-hand side converges. Due to the fact that f is continuous for all $x \in \mathbb{R}$ that are not odd integers, i.e. $x \neq 2k+1$ for all $k \in \mathbb{Z}$, one has that $f(x) = \frac{1}{2}(f(x^-) + f(x^+))$ in this case. On the other hand, we find that

$$\frac{1}{2}(f(2k+1)^-) + f(2k+1)^+) = \frac{1}{2}(1+0) = \frac{1}{2} = f(2k+1)$$

for all $k \in \mathbb{Z}$. Summing up, we have for all $x \in \mathbb{R}$ that

$$f(x) = \frac{1}{2}(f(x^-) + f(x^+)) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{-ik\pi x}.$$

Along with the Euler formula we find that

$$\begin{aligned} f(x) &= \hat{f}_0 + \sum_{k=1}^{\infty} (\hat{f}_k + \hat{f}_{-k})\cos(k\pi x) + i(\hat{f}_k - \hat{f}_{-k})\sin(k\pi x) \\ &= a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x), \qquad x \in \mathbb{R}, \end{aligned}$$

with $a_0 = \hat{f}_0$, $a_k = \hat{f}_k + \hat{f}_{-k}$ and $b_k = i(\hat{f}_k - \hat{f}_{-k})$ for $k \in \mathbb{N}$. Using the calculations in b) gives that $a_0 = \frac{1}{4}$ and for $k \in \mathbb{N}$,

$$a_k = \begin{cases} 0 & \text{if } k \text{ is even}, \\ -\frac{2}{k^2\pi^2} & \text{if } k \text{ is odd}, \end{cases}$$

and

$$b_k = \begin{cases} -\frac{1}{k\pi} & \text{if } k \text{ is even}, \\ \frac{1}{k\pi} & \text{if } k \text{ is odd}. \end{cases}$$

Finally, we obtain the Fourier sine and cosine series representation

$$f(x) = \frac{1}{4} - \sum_{k=1,\, k\,\mathrm{odd}}^{\infty} \frac{2}{k^2\pi^2}\cos(k\pi x) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\pi}\sin(k\pi x), \qquad x \in \mathbb{R}. \tag{6}$$

d) We set $x = 0$ in (6) to find that

$$0 = f(0) = \frac{1}{4} - \sum_{k=1, k\,\mathrm{odd}}^{\infty} \frac{2}{k^2\pi^2},$$

which can be rewritten as

$$\sum_{k=1,\, k\,\mathrm{odd}}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{8}.$$

Hence, the sought value of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$ is $\frac{\pi^2}{8}$.

Problem 4.

a) Following the separation of variables method, we assume that the solution u to (2) has the form

$$u(x,t) = X(x)T(t), \qquad x \in \mathbb{R},\ t \geq 0,$$

with functions $X : \mathbb{R} \to \mathbb{C}$ and $T : [0,\infty) \to \mathbb{C}$. Plugging this ansatz into (2) gives that

$$\frac{X'(x)}{X(x)} = -\frac{\dot{T}(t)}{T(t)}, \quad x \in \mathbb{R},\ t > 0.$$

Since the right-hand side of this equation depends only on x and the left-hand side only on t, there exists a constant $\mu \in \mathbb{C}$ such that

$$\begin{aligned} X'(x) &= \mu X(x), \qquad & x \in \mathbb{R}, \\ \dot{T}(t) &= -\mu T(t), \qquad & t > 0. \end{aligned}$$

The general (complex) solution to the differential equation $X' = \mu X$ is given by $X(x) = c_1 e^{\mu x}$ for $x \in \mathbb{R}$ with a constant $c_1 \in \mathbb{C}$. Similarly, $T(t) = c_2 e^{-\mu t}$ for $t > 0$ with a constant $c_2 \in \mathbb{C}$ is the general (complex) solution to $\dot{T} = -\mu T$.
This implies that $u(x,t) = ce^{\mu(x-t)}$ for $x \in \mathbb{R}$ and $t \geq 0$ with a constant $c \in \mathbb{C}$. The initial condition $u(x,0) = e^{2x}$ requires that $ce^{\mu x} = e^{2x}$ for all $x \in \mathbb{R}$, which means that $c = 1$ and $\mu = 2$.
Summing up, one finds that $u(x,t) = e^{2(x-t)}$ for $x \in \mathbb{R}$ and $t \geq 0$ is the desired solution.

b) Since for all $x \in \mathbb{R}$ and $t > 0$,

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}g(x-t) = -g'(x-t)$$

and

$$\frac{\partial}{\partial x}u(x,t) = \frac{\partial}{\partial x}g(x-t) = g'(x-t),$$

it follows that $u(x,t) = g(x-t)$ indeed satisfies (2). Moreover, $u(x,0) = g(x-0) = g(x)$ for $x \in \mathbb{R}$, so that also the initial condition (3) is fulfilled.

c) No. In fact, the function u defined by $u(x,t) = \sin(x-t)$ for $x \in \mathbb{R}$ and $t \geq 0$ does not have the multiplicative structure assumed in the separation of variables approach. This can be proved by a contradiction argument as follows. Assume that $u(x,t) = \sin(x-t) = X(x)T(t)$ for $x \in \mathbb{R}$ and $t \geq 0$ with functions $X : \mathbb{R} \to \mathbb{C}$ and $T : [0,\infty) \to \mathbb{C}$. Then

$$\begin{aligned} \sin(x) &= T(0)X(x), \\ \cos(x) &= -\sin(x - \tfrac{\pi}{2}) = -T(\tfrac{\pi}{2})X(x) \end{aligned}$$

for all $x \in \mathbb{R}$. Since the sine and cosine function do not just vary by a scalar factor, consequently, $T(0) = T(\frac{\pi}{2}) = 0$. This yields $\sin(x) = \cos(x) = 0$ for all $x \in \mathbb{R}$, which is a contradiction.

Problem 5.

a) We recall that with $\mathcal{F}$ denoting the Fourier transformation,

$$(\mathcal{F}(v'))(s) = \widehat{v'}(s) = is\hat{v}(s) \quad \text{and} \quad (\mathcal{F}(v''))(s) = \widehat{v''}(s) = (is)^2\hat{v}(s) = -s^2\hat{v}(s)$$

for $s \in \mathbb{R}$. Hence, applying $\mathcal{F}$ to (5) results in

$$-s^2\hat{v}(s) + 4is\hat{v}(s) + 3\hat{v}(s) = \hat{f}(s), \qquad s \in \mathbb{R}.$$

We solve for $\hat{v}$ to obtain

$$\hat{v}(s) = \frac{\hat{f}(s)}{-s^2 + 4is + 3}$$

for $s \in \mathbb{R}$.

b) We calculate that

$$(\mathcal{F}g)(s) = \hat{g}(s) = \frac{1}{2}\int_0^\infty (e^{-t} - e^{-3t})e^{-ist}\, dt = \frac{1}{2}\Big(\frac{1}{1+is} - \frac{1}{3+is}\Big) = \frac{1}{-s^2+4is+3}.$$

c) By definition of the convolution product $f * g$ and the function f one has that

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\, dt = 2\int_{x-1}^{x} g(t)\, dt.$$

If $x \leq 0$, then

$$(f*g)(x) = 2\int_{x-1}^{x} g(t)dt = 0.$$

For $x \in (0,1)$,

$$(f*g)(x) = 2\int_0^x g(t)\, dt = \int_0^x e^{-t} - e^{-3t}\, dt = \frac{1}{3}e^{-3x} - e^{-x} + \frac{2}{3}.$$

and for $x \geq 1$,

$$\begin{aligned} (f*g)(x) = 2\int_{x-1}^{x} g(t)dt &= -\frac{1}{3}e^{-3x+3} + e^{-x+1} + \frac{1}{3}e^{-3x} - e^{-x} \\ &= \frac{1}{3}(1-e^3)e^{-3x} + (e-1)e^{-x}. \end{aligned}$$

Summing up,

$$(f * g)(x) = \begin{cases} \frac{1}{3}e^{-3x} - e^{-x} + \frac{2}{3} & \text{if } x \in (0,1), \\ \frac{1}{3}(1-e^3)e^{-3x} + (e-1)e^{-x} & \text{if } x \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

d) In view of a) and b), we have that

$$\hat{v} = \hat{f}\hat{g} = \widehat{f * g},$$

and Fourier inversion implies that

$$v = f * g.$$

For an explicit expression for this convolution product see c). Hence, $\bar{v} = f * g$ is a particular solution to (5).

e) The general solution to an inhomogeneous linear differential equation can be obtained by adding a particular solution to a solution of the corresponding homogeneous equation, which in the case of (5) is

$$v'' + 4v' + 3v = 0. \tag{7}$$

Since the roots of $\lambda^2 + 4\lambda + 3 = 0$ are exactly $\lambda = -1$ and $\lambda = -3$, the general (real) solution v_{hom} to (7) is given by

$$v_{\text{hom}}(t) = \alpha e^{-t} + \beta e^{-3t} \qquad \text{for } t \in \mathbb{R},$$

with constants $\alpha, \beta \in \mathbb{R}$. Hence, the general solution to (5) is $v = \bar{v} + v_{\text{hom}}$ with $\bar{v}$ as in d).