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**MATHEMATICAL METHODS 3 (WT3)**
**Midterm exam**


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*Please solve each exercise on a separate sheet of paper and write your name on all of them!  
No documents nor electronic devices allowed. All answers must be justified.*

**Exercise 1** (4 points).

a) Find a function of the form  $y_1 = a + bx^2$  (for some real numbers  $a$  and  $b$  to be determined), which solves the following ODE

$$(1) \quad x^2 y'' - 2xy' + 2y = 0, \quad x > 0.$$

b) Find a second solution  $y_2$ , such that  $y_1$  and  $y_2$  are linearly independent. Find all real solutions of (1).

c) Using the method of variation of parameters, find the general real solution of the ODE

$$x^2 y'' - 2xy' + 2y = x^2 + 1, \quad x > 0.$$

d) Solve the initial value problem

$$\begin{cases} x^2 y'' - 2xy' + 2y = x^2 + 1, & x > 0, \\ y(1) = 2, & y'(1) = 1. \end{cases}$$


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**Exercise 2** (3 points). Let  $A$  be the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

a) Find the eigenvalues of  $A$ .

b) Find two linearly independent eigenvectors of  $A$ .

c) Find all complex solutions of the system of ODEs

$$\begin{cases} y_1' = y_1 - y_2, \\ y_2' = y_1 + y_2. \end{cases}$$

Then, find all real solutions of the above system of ODEs.

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**Exercise 3** (3 points). We consider the differential equation

$$(2) \quad y'' - 2xy' + 2y = 0$$

We are looking for solutions of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^k.$$

a) Let  $y$  be as above. Write  $y'$  and  $y''$  as power series. Write  $y'' - 2xy' + 2y$  as a power series.

b) Show that  $y$  is a solution of (2) if and only if

$$a_{k+2} = +2 \frac{(k-1)}{(k+1)(k+2)} a_k, \quad k = 0, 1, 2, \dots$$

c) Find the solution such that  $a_0 = 0$  and  $a_1 = 3$ .

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