Utrecht University 2024/25

Mathematical Methods 3 (WT3)

Exam 1

Please solve each exercise on a separate sheet of paper and write your name on each. No documents/electronic devices allowed. Answers must be justified. Each question is worth 0.5 p.

Good luck and *veel succes*!

Exercise 1 (3.5 points). Let A be the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}.$$

a) Compute the characteristic polynomial p_A of A.
b) Find all complex eigenvalues of A.
c) Find two eigenvectors u, v of A (such that one is not a multiple of the other).
d) Find all real solutions of the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -2x + 3y. \end{cases}$$

e) Find all real solutions of the ODE

$$\ddot{z} - 3\dot{z} + 2z = 0.$$

Then, find the particular solution such that $z(0) = 1$ and $\dot{z}(0) = 2$.
f) Find a particular solution of the ODE

$$\ddot{z} - 3\dot{z} + 2z = \sin t.$$

of the form $z_p(t) = a\cos t + b\sin t$, with a, b two constants to be determined. Then, give the general real solution of that ODE.
g) Using the power series method find a particular solution of the ODE

$$\ddot{z} - 3\dot{z} + 2z = (t-1)\sum_{n=0}^{\infty} \frac{t^n}{n!}$$

of the form $z(t) = c\sum_{n=0}^{\infty} \frac{t^{n+2}}{n!}$ for some constant c to be determined.

Exercise 2 (4.5 points). We consider the partial differential equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < 2, \; 0 < y < 1,$$

where the unknown is a function of two variables $u(x, y)$. We also consider the boundary conditions

$$(2) \qquad u(x, 0) = 0 = \frac{\partial u}{\partial y}(x, 1) \;\; \text{for } 0 < x < 2,$$

and

$$(3) \qquad u(0, y) = 0, \quad u(2, y) = y(2 - y) \;\; \text{for } 0 < y < 1.$$

a) Substitute a function of the form $X(x)Y(y)$ into (1) and write an equivalent pair of ODEs for X and Y.

b) Find all solutions of (1) of the form $X(x)Y(y)$ and satisfying the two boundary conditions (2) and the first boundary condition in (3) (such that $X(x) \neq 0$ and $Y(y) \neq 0$ for $0 < x < 2$ and $0 < y < 1$).

c) Let f be the function periodic of period 4 such that

$$f(y) = \begin{cases} y(2-y) \text{ if } 0 \leqslant y < 2\,, \\ y(2+y) \text{ if } -2 < y < 0\,. \end{cases}$$

Is f odd, or even, on the interval $-2 < y < 2$? Justify.

d) Calculate the integral $\int_0^2 y(2-y)\sin(\frac{n\pi y}{2})dy$ for $n = 0, 1, 2, \ldots$.

e) Show that

$$(4) \qquad y(2-y) = c\sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin\left(\frac{(2m-1)\pi y}{2}\right) \quad \text{for } \ 0 < y < 2.$$

where c is a constant to be determined.

f) Is the function $g = \sin(3\pi y)$ orthogonal to f for the inner product given by

$$(u, v) = \int_{-2}^{2} u(y)v(y)dy?$$

Is the function $h = \sin(\frac{5}{2}\pi y)$ orthogonal to f? Justify your answers.

g) Find the solution of (1) satisfying the boundary conditions (2) and (3), expressed as a sum of series.

h) Using **e)** show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}.$$

i) Determine the radius of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{(2k-1)^3}.$$

Exercise 3 (2 points). Let

$$f(x) = e^{-|x|}\cos x.$$

a) Express f in terms of complex exponentials. Compute the Fourier transform $\widehat{f}$ of f.

b) Compute the Fourier transform $\widehat{g}$ of the function $g(x) = e^{-|2x+1|}\cos(2x+1)$ and the Fourier transform $\widehat{f * g}$ of the function $f * g$.

c) Suppose that h is an absolutely integrable function on $\mathbb{R}$ that solves the ODE

$$-h'' + 2h = f.$$

Find the Fourier transform $\widehat{h}$ of h.

d) Compute the expression

$$\int_{-\infty}^{\infty} f(3x)(\delta(x+\pi) + \delta(x-\pi))dx = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f(3x)(\delta_\varepsilon(x+\pi) + \delta_\varepsilon(x-\pi))dx.$$

where δ_ε is an approximation of the Dirac delta function δ.

Here is a reminder of various formulas and conventions that you might need:

- (Complex exponential)
$$e^{ix} = \cos x + i\sin x, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$
- (Special values of trigonometric functions)
$$\text{For all } n \in \mathbb{Z},\ \ \sin(n\pi) = 0 \text{ and } \cos(n\pi) = \begin{cases} 1 \text{ if } n \text{ is even,} \\ -1 \text{ if } n \text{ is odd.} \end{cases}$$
- (Wronskian)
$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$
- (Fourier series for $2L$-periodic functions)
$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right) \right),$$
where
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right)\,dx.$$
- (Complex Fourier series for T-periodic functions)
$$\hat{f}_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)e^{-ik\omega x}dx \text{ for } k \in \mathbb{Z}, \text{ where } \omega = \frac{2\pi}{T},$$
and then
$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{ik\omega x}.$$
- (Fourier transform, inverse Fourier transform)
$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx, \quad (\mathcal{F}^{-1}g)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)e^{ix\xi}d\xi$$
- (Convolution)
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad \widehat{f * g} = \sqrt{2\pi}\,\widehat{f}\,\widehat{g}$$
- (Dirac delta function)
An example of an approximation of the Dirac delta function is δ_ε, defined for $\varepsilon > 0$ by:
$$\delta_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon} \text{ if } -\varepsilon \leqslant x \leqslant \varepsilon \\ 0 \text{ otherwise.} \end{cases}$$
By definition,
$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(x)\delta_\varepsilon(x)dx.$$