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Midterm Ergodic Theory 2009

Due Date: April 21, 2009- **You are not allowed to discuss this exam with your fellow students**

1. Let $(X, \mathcal{F}, \mu, T)$ be a measure preserving system, and assume T is ergodic. Let f be a meausrable real valued function on X such that $\mu(\{x \in X : f(x) = 0\}) > 0$. Define g on X by $g(x) = f(x) - f(Tx)$. Prove that

$$\mu(\{x : \sum_{i=0}^{n-1} g(T^i x) = f(x) \text{ for infinitely many } n \geq 1\}) = 1.$$

2. Let $\theta \in (0, 1)$ be irrational.

   (a) Consider the probability space $([0, 1), \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra and λ is Lebesgue measure restricted to $[0, 1)$. Let $T : [0, 1) \to [0, 1)$ be translation by $\theta \in (0, 1)$, i.e. $Tx = x + \theta \mod 1$. Determine explicitly the induced transformation T_A of T on the interval $A = [0, \theta)$.

   (b) Consider the probability space $([0, 1]\times[0, 1], \mathcal{B}\times\mathcal{B}, \lambda\times\lambda)$, where $\mathcal{B}\times\mathcal{B}$ is the two-dimesional Borel σ-algebra and $\lambda \times \lambda$ is the two-dimensional Lebesgue measure restricted to $[0, 1] \times [0, 1]$. Prove that the transformation $S : [0, 1] \times [0, 1] \to [0, 1]\times[0, 1]$ given by $S(x, y) = (x+\theta \mod 1, x+y \mod 1)$ is measure preserving and ergodic with respect to $\lambda \times \lambda$.

   (Hint: The Fourier series $\sum_{n,m} c_{n,m} e^{2\pi i(nx+my)}$ of a function $f \in L^2([0, 1]\times[0, 1], \mathcal{B}\times \mathcal{B}, \lambda \times \lambda)$ satisfies $\sum_{n,m} |c_{n,m}|^2 < \infty$.

3. Consider $([0, 1), \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Lebesgue σ-algebra, and λ is Lebesgue measure. Let $T : [0, 1) \to [0, 1)$ be defined by

$$Tx = \begin{cases} n(n+1)x - n, & x \in [\frac{1}{n+1}, \frac{1}{n}), \\ 0, & x = 0. \end{cases}$$

   Define $a_1 : [0, 1) \to [2, \infty]$ by

$$a_1 = a_1(x) = \begin{cases} n+1 & \text{if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right), \; n \geq 1 \\ \infty & \text{if } x = 0. \end{cases}$$

   For $n \geq 1$, let $a_n = a_n(x) = a_1(T^{n-1}x)$.

   (a) Show that T is measure preserving with respect to Lebesgue measure λ.

(b) Show that for λ a.e. x there exists a sequence $a_1, a_2, \cdots$ of positive integers such that $a_i \geq 2$ for all $i \geq 1$, and

$$x = \frac{1}{a_1} + \frac{1}{a_1(a_1-1)a_2} + \cdots + \frac{1}{a_1(a_1-1)\cdots a_{k-1}(a_{k-1}-1)a_k} + \cdots .$$

(c) Consider the dynamical system $(X, \mathcal{F}, \mu, S)$, where $X = \{2, 3, \cdots\}^{\mathbb{N}}$, $\mathcal{F}$ the σ-algebra generated by the cylinder sets, S the left shift on X, and μ the product measure with $\mu(\{x : x_1 = j\}) = \dfrac{1}{j(j-1)}$. Show that $([0,1), \mathcal{B}, \lambda, T)$ and $(X, \mathcal{F}, \mu, S)$ are isomorphic.

(d) Show that T is strongly mixing.

(e) Consider the product space $([0,1) \times [0,1), \mathcal{B} \times \mathcal{B}, \lambda \times \lambda)$. Define the transformation $\mathcal{T} : [0,1) \times [0,1) \to [0,1) \times [0,1)$ by

$$\mathcal{T}(x,y) = \begin{cases} (Tx, \dfrac{y+n}{n(n+1)}) & \text{if } x \in \left[\dfrac{1}{n+1}, \dfrac{1}{n}\right) \\ (0,0) & \text{if } x = 0. \end{cases}$$

Show that $([0,1)\times[0,1), \mathcal{B}\times\mathcal{B}, \lambda\times\lambda, \mathcal{T})$ is a natural extension of $([0,1), \mathcal{B}, \lambda, T)$.

4. Consider $([0,1), \mathcal{B})$, where $\mathcal{B}$ is the Lebesgue σ-algebra. Let $T : [0,1) \to [0,1)$ be the *Continued fraction* transformation, i.e., $T0 = 0$ and for $x \neq 0$

$$Tx = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor.$$

It is well-known that T is measure preserving and ergodic with respect to the *Gauss-measure* μ given by

$$\mu(B) = \int_B \frac{1}{\log 2} \frac{1}{1+x} dx$$

for every Lebesque set B. For each $x \in [0,1)$ consider the sequence of digits of x defined by $a_n(x) = a_n = \lfloor \dfrac{1}{T^{n-1}x} \rfloor$. Let λ denote the normalized Lebesgue measure on $[0,1)$.

(a) Show that $\displaystyle\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty$ λ a.e.

(b) Show that

$$\lim_{n\to\infty} (a_1 a_2 \ldots a_n)^{1/n} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}}$$

λ a.e.