Final Exam Ergodic Theory

January 6, 2015 - **You are allowed to use the lecture notes, but no other documents or calculators.**

1. (a) Let $F : [0,1] \to [0,1]$ be the Farey map (see Figure 1), which is given by

$$Fx = \begin{cases} \frac{x}{1-x}, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ \frac{1-x}{x}, & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

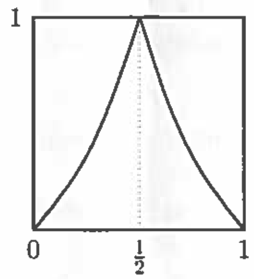

Figure 1: The Farey map F.

Show that F is measure preserving with respect to the measure μ given by

$$\mu(A) = \int_A \frac{1}{x}\, d\lambda,$$

where A is a Lebesgue set and λ is the Lebesgue measure.

(b) Give an example of a Lebesgue set A for which $\mu\big(F(A)\big) \neq \mu(A)$. Justify your answer.

(c) For each $x \in (0,1]$, define the number $n(x)$ by

$$n(x) = \min\left\{n \geq 0 \,:\, F^n x \in \left(\frac{1}{2}, 1\right]\right\}.$$

Show that $n(x) = n - 1$ for all $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$.

(d) Let T be the Gauss map, given by $Tx = \frac{1}{x} \pmod 1$. Using the notation from part (c), show that for each $x \in (0,1)$,

$$F^{n(x)+1} x = Tx.$$

2. Given a measure preserving system $(X, \mathcal{F}, \mu, T)$, show that $T \times T$ is ergodic with respect to $\mu \times \mu$ if and only if $T \times T \times T$ is ergodic with respect to $\mu \times \mu \times \mu$.

3. (a) Let $(X, \mathcal{F}, \mu, T)$ be a measure preserving and ergodic system, and $f \in L^1(\mu)$. Show that for any $\alpha \in \mathbb{R}$ the set

$$X_\alpha = \left\{ x \in X \, : \, \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \alpha \right\}$$

satisfies $\mu(X_\alpha) \in \{0, 1\}$.

(b) Let $\theta \in [0, 1) \backslash \mathbb{Q}$ and let $S^1 = \{e^{2\pi i \alpha} \, : \, \alpha \in [0, 1)\}$. Consider the dynamical system $(S^1, \mathcal{B}, \lambda, R_\theta)$, where $\mathcal{B}$ is the Lebesgue σ-algebra, λ is the normalised Lebesgue measure on S^1 and $R_\theta : S^1 \to S^1$ is given by $R_\theta(e^{2\pi i \alpha}) = e^{2\pi i(\alpha + \theta)}$. Let $f \in C([0, 1])$. Using the notation of part (a), determine for each $\alpha \in \mathbb{R}$ the set X_α explicitly.

$f \in C(S^1)$

4. Let $(X, \mathcal{F}, \mu, T)$ be a measure preserving dynamical system. Let $k > 0$.

(a) Show that for any finite partition α of X one has $h_\mu\left(\bigvee_{i=0}^{k-1} T^{-i}\alpha, T^k \right) = k h_\mu(\alpha, T)$.

(b) Prove that $k h_\mu(T) \leq h_\mu(T^k)$.

(c) Prove that $h_\mu(\alpha, T^k) \leq k h_\mu(\alpha, T)$.

(d) Prove that $h_\mu(T^k) = k h_\mu(T)$.