Universiteit Utrecht
Mathematisch Instituut

Boedapestlaan 6
3584 CD Utrecht

Midterm Ergodic Theory

Due Date: November 22, 2004

1. Consider $([0,1),\mathcal{B})$, where $\mathcal{B}$ is the Lebesgue σ-algebra. Let $T:[0,1)\to[0,1)$ be the *Continued fraction* transformation, i.e., $T0=0$ and for $x\neq 0$
$$Tx=\frac{1}{x}-\lfloor\frac{1}{x}\rfloor.$$
It is well-known that T is measure preseving and ergodic with respect to the *Gauss-measure* μ given by
$$\mu(B)=\int_B\frac{1}{\log 2}\frac{1}{1+x}dx$$
for every Lebesque set B. For each $x\in[0,1)$ consider the sequence of digits of x defined by $a_n(x)=a_n=\lfloor\frac{1}{T^{n-1}x}\rfloor$. Show that
$$\lim_{n\to\infty}(a_1a_2\dots a_n)^{1/n}=\prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}}$$
for Lebesque a.e. x.

2. Let T be a measure preserving and ergodic transformation on the probability space $(X,\mathcal{F},\mu)$. Let $g\in L^1(X,\mathcal{F},\mu)$ be real valued, and $A=\{x\in X:g(x)=0\}$. Define $f:X\to\mathbb{R}$ by $f(x)=g(x)-g(Tx)$ and set $f_n(x)=\sum_{i=0}^{n-1}f(T^ix)$, $n\geq 1$. Show that if $\mu(A)>0$, then for μ a.e. $x\in X$ there exist infinitely many positive integers n such that $f_n(x)=g(x)$.

3. Let $(X,\mathcal{F},\mu)$ be a probability space, and let $T:X\to X$ measure preserving and ergodic. Consider the probability space $(Y,\mathcal{G},\nu)$, where
$$Y=X\times\{0\}\cup X\times\{1\},$$
$\mathcal{G}$ the σ-algebra generated by sets of the form $A\times\{i\}$ with $A\in\mathcal{F}$, $i=0,1$, and ν the measure given by $\nu(A\times\{i\})=\frac{1}{2}\mu(A)$. Define $S:Y\to Y$ by $S(x,0)=(x,1)$ and $S(x,1)=(Tx,0)$.

   (a) Show that S is measure preserving and ergodic with respect to ν.

   (b) Show that S is not strongly mixing.

4. Let $(X, \mathcal{F}, \mu)$ be a probability space, and $T : X \to X$ a measure preserving transformation. Consider the transformation $T \times T$ defined on $(X \times X, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ by $(T \times T)(x, y) = (Tx, Ty)$.

   (i) Show that $T \times T$ is measure preserving with respect to $\mu \times \mu$.

   (ii) Show that T is strongly mixing with respect to μ if and only if $T \times T$ is strongly mixing with respect to $\mu \times \mu$.

   (iii) Show that if $T = T_\theta = x + \theta \pmod 1$ is an irrational rotation on $[0, 1)$, then T_θ is **not** weakly mixing with respect to Lebesgue measure λ on $[0, 1)$..

5. Let λ be the normalized Lebesque measure on $([0, 1), \mathcal{B})$, where $\mathcal{B}$ is the Lebesgue σ-algebra. Consider the transformation $T : [0, 1) \to [0, 1)$ given by

$$Tx = \begin{cases} 3x & 0 \le x < 1/3 \\ \frac{3}{2}x - \frac{1}{2} & 1/3 \le x < 1. \end{cases}$$

For $x \in [0, 1)$ let

$$s_1(x) = \begin{cases} 3 & 0 \le x < 1/3 \\ \frac{3}{2} & 1/3 \le x < 1, \end{cases}$$

$$h_1(x) = \begin{cases} 0 & 0 \le x < 1/3 \\ \frac{1}{2} & 1/3 \le x < 1, \end{cases}$$

and

$$a_1(x) = \begin{cases} 0 & 0 \le x < 1/3 \\ 1 & 1/3 \le x < 1. \end{cases}$$

Let $s_n = s_n(x) = s_1(T^{n-1}x)$, $h_n = h_n(x) = h_1(T^{n-1}x)$ and $a_n = h_n(x) = a_1(T^{n-1}x)$ for $n \ge 1$.

   (a) Show that for any $x \in [0, 1)$ one has $x = \displaystyle\sum_{k=1}^{\infty} \frac{h_k}{s_1 s_2 \cdots s_k}$.

   (b) Show that T is measure preseving and ergodic with respect to the measure λ.

   (c) Show that for each $n \ge 1$ and any sequence $i_1, i_2, \ldots, i_n \in \{0, 1\}$ one has

$$\lambda\left(\{x \in [0, 1) : a_1(x) = i_1, a_2(x) = i_2, \ldots, a_n(x) = i_n\}\right) = \frac{2^k}{3^n},$$

   where $k = \#\{1 \le j \le n : i_j = 1\}$.

   (c) Show that $a_1, a_2, \ldots,$ is a sequence of independent identically distributed random variables on $[0, 1)$.