DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UU.
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Ergodic theory (WISM464)
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Question 1

Consider $([0,1),\mathcal{B})$, where $\mathcal{B}$ is the Lebesgue σ-algebra. Let $T:[0,1)\to[0,1)$ be the *continued fraction* transformation, i.e., $T0=0$ and for $x\neq 0$,

$$Tx=\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor.$$

It is well-known that T is measure preserving and ergodic with respect to the *Gauss-measure* μ given by

$$\mu(B)=\int_B\frac{1}{\log 2}\frac{1}{1+x}\,\mathrm{d}x$$

for every Lebesque set B. For each $x\in[0,1)$ consider the sequence of digits of x defined by $x_n(x)=a_n=\left\lfloor\frac{1}{T^{n-1}x}\right\rfloor$. Let λ denote the normalized Lebesgue measure on $[0,1)$.

a) Show that $\lim\limits_{n\to\infty}\dfrac{a_1+a_2+\cdots+a_n}{n}=\infty$ λ a.e.

b) Show that

$$\lim_{n\to\infty}(a_1a_2\ldots a_n)^{1/n}=\prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}}$$

λ a.e.

Question 2

Let $(X,\mathcal{F},\mu)$ be a probability space, and $T:X\to X$ a measure preserving transformation. Let $A\in\mathcal{F}$ with $\mu(A)>0$. For $x\in A$ let $n(x)$ be the first return time of x to A, and μ_A the induced measure on the σ-algebra $\mathcal{F}\cap A$ on A. Consider the induced transformation T_A of T on A given by $T_Ax=T^{n(x)}x$.

a) Show that if T_A is ergodic and $\mu\left(\bigcup_{k\geq 1}T^{-k}A\right)=1$, then T is ergodic.

b) Assume further that T is invertible and ergodic.

(i) Show that

$$\int_A n(x)\,\mathrm{d}\mu=1.$$

(ii) Prove that

$$\mu_A\left(\{x\in A:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}n(T_A^i(x))=\frac{1}{\mu(A)}\}\right)=1.$$

Question 3

Let $(X, \mathcal{F}, \mu)$ be a probability space, and $T : X \to X$ a measure preserving transformation. Let $f \in L^1(X, \mathcal{F}, \mu)$.

a) Show that if $f(Tx) \leq f(X)$ μ a.e., then $f(x) = f(Tx)$ μ a.e.

b) Show that $\lim_{n\to\infty} \dfrac{f(T^n x)}{n} = 0$ μ a.e.

Question 4

Let $(X, \mathcal{F}, \mu)$ be a probability space, and $T : X \to X$ a measure preserving transformation. Consider the transformation $T \times T$ defined on $(X \times X, \mathcal{F} \times \mathcal{F}, \mu \times \mu)$ by $(T \times T)(x, y) = (Tx, Ty)$.

a) Show that T is strongly mixing with respect to μ if and only if $T \times T$ is strongly mixing with respect $\mu \times \mu$.

b) Show that T is weakly mixing with respect to μ if and only if $T \times T$ is ergodic with respect to $\mu \times \mu$.

c) Show that $T = T_\theta = x + \theta \pmod 1$ is an irrational rotation on $[0, 1)$, then T_θ is *not* weakly mixing with respect to $\lambda \times \lambda$ where λ is the normalized Lebesgue measure on $[0, 1)$.

Question 5

Let λ be the normalized Lebesgue measure on $([0, 1), \mathcal{B})$ where $\mathcal{B}$ is the Lebesgue σ-algebra. Consider the transformation $T : [0, 1) \to [0, 1)$ given by

$$Tx = \begin{cases} 3x & 0 \leq x < 1/3 \\ \frac{3}{2}x - \frac{1}{2} & 1/3 \leq x < 1. \end{cases}$$

For $x \in [0, 1)$ let

$$s_1(x) = \begin{cases} 3 & 0 \leq x < 1/3 \\ \frac{3}{2} & 1/3 \leq x < 1. \end{cases}$$

$$h_1(x) = \begin{cases} 0 & 0 \leq x < 1/3 \\ \frac{1}{2} & 0 \leq x < 1. \end{cases}$$

and

$$a_1(x) = \begin{cases} 0 & 0 \leq x < 1/3 \\ 1 & 1/3 \leq x < 1. \end{cases}$$

Let $s_n = s_n(x) = s_1(T^{n-1}x$, $h_n = h_n(x) = h_1(T^{n-1}x)$ and $a_n = a_n(x) = a_1(T^{n-1}x)$ for $n \geq 1$.

a) Show that for any $x \in [0, 1)$ one has

$$x = \sum_{k=1}^{\infty} \frac{h_k}{s_1 s_2 \cdots s_k}.$$

b) Show that T is measure preserving and ergodic with respect to the measure λ.

c) Shwo that for each $n \geq 1$ and any sequence $i_1, i_2, \ldots i_n \in \{0, 1\}$ one has

$$\lambda(\{x \in [0, 1) : a_1(x) = i_1, a_2(x) = i_2, \ldots a_n(x) = i_n\}) = \frac{2^k}{3^n},$$

where $k = \#\{1 \leq j \leq n : i_j = 1\}$.