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In 2006/2007, the course WISM464 was given by Dr. K. Dajani.

Ergodic Theory (WISM464)
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You are not allowed to discuss this exam with your fellow students.

Question 1

Consider $([0,1),\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Lebesque σ-algebra, and λ is Lebesque measure. Let $T:[0,1)\to[0,1)$ be defined by

$$Tx=\begin{cases} n(n+1)x-n & \text{if } x\in\left[\frac{1}{n+1},\frac{1}{n}\right) \\ 0 & \text{if } x=0 \end{cases}$$

Define $a_1:[0,1)\to[2,\infty]$ by

$$a_1=a_1(x)=\begin{cases} n+1 & \text{if } x\in\left[\frac{1}{n+1},\frac{1}{n}\right),\ n\geq 1 \\ \infty & \text{if } x=0 \end{cases}$$

For $n\geq 1$, let $a_n=a_n(x)a_1\left(T^{n-1}x\right)$.

a) Show that T is measure preserving with respect to Lebesgue measure λ.

b) Show that for λ a.e. x there exists a sequence $a_1,a_2,\ldots$ of positive integers such that $a_1\geq 2$ for all $i\geq 1$, and

$$x=\frac{1}{a_1}+\frac{1}{a_1(a_1-1)a_2}+\ldots+\frac{1}{a_1(a_1-1)\ldots a_{k-1}(a_{k-1}-1)a_k}+\ldots$$

c) Consider the dynamical system $(X,\mathcal{F},\mu,S)$ where $X=\{2,3,\ldots\}^{\mathbb{N}},\mathcal{F}$ the σ-algebra generated by the cylinder sets, S the left shift on X, and μ the product measure with $\mu\left(\{x:x_1=j\}\right)=\frac{1}{j(j-1)}$. Show that $([0,1),\mathcal{B},\lambda,T)$ and $(X,\mathcal{F},\mu,S)$ are isomorphic. Conclude that T is a strongly mixing transformation.

d) Consider the product space $([0,1)\times[0,1),\mathcal{B}\times\mathcal{B},\lambda\times\lambda)$. Define the transformation $\mathcal{T}:[0,1)\times[0,1)\to[0,1)\times[0,1)$ by

$$\mathcal{T}(x,y)=\begin{cases} \left(Tx,\frac{y+n}{n(n+1)}\right) & \text{if } x\in\left[\frac{1}{n+1},\frac{1}{n}\right) \\ (0,0) & \text{if } x=0 \end{cases}$$

1. Show that $\mathcal{T}$ is measurable and measure preserving with respect to $\lambda\times\lambda$. Prove also that $\mathcal{T}$ is one-to-one and onto $\lambda\times\lambda$ a.e.
2. Show that $([0,1]\times[0,1),\mathcal{B}\times\mathcal{B},\lambda\times\lambda,\mathcal{T})$ is a natural extension of $([0,1),\mathcal{B},\lambda,T)$

Question 2

Let $(X,\mathcal{F},\mu)$ be a probability space and $T:X\to X$ a measure preserving transformation. Let $k>0$.

a) Show that for any finit partition α of X one has $h_\mu\left(\bigvee_{i=0}^{k-1}\alpha,T^k\right)=kh_\mu(\alpha,T)$.

b) Prove that $kh_\mu(T)\leq h_\mu(T^k)$.

c) Prove that $h_\mu(\alpha,T^k)\leq kh_\mu(\alpha,T)$.

d) Prove that $h_\mu(T^k)=kh_\mu(T)$.

Question 3

Let X be a compact metric space, (B) the Borel σ-algebra on X and $T : X \to X$ a uniquely ergodic continuous transformation. Let μ be the unique ergodic measure, and assume that $\mu(G) > 0$ for all non-empty open sets $G \subseteq X$.

a) Show that for each non-empty open subset G of X there exists a continuous function $f \in C(X)$, and a closed subset F of G of positive μ measure such that $f(x) = 1$ for $x \in F$, $f(x).0$ for $x \in G$ and $f(x) = 0$ for $x \in X\ G$

b) Show that for each $x \in X$ and for every non-empty open set $G \subseteq X$, there exissts $n \geq 0$ such that $T^n x \in G$. Conclude that $\{T^n x : n \geq 0\}$ is dense in X.

Question 4

Let X be a compact metric space, and (B) the Borel σ-algebra on X and $T : X \to X$ be a continuous transformation. Let $N \geq 1$ and $x \in X$.

a) Show that $T^N x = x$ if and only if $\frac{1}{N}\sum_{i=0}^{N-1} \delta_{T^i x} \in M(X,T)$. ($\delta_y$ is the Dirac measure concentrated at the point y.)

b) Suppose $X = 1, 2, \ldots, N$ and $Ti = i + 1(mod(N))$. Show that T is uniquely ergodic. Determine the uniqueergodic measure.

Question 5

use the Shannon-McMillan-Breiman Theorem (and the Ergodic Theorem if necessary) in order to show that

a) $h_\mu(T) = log\beta$, where $\beta = \frac{1+\sqrt{5}}{2}$, T the β-transformation defined on $([0,1), \mathcal{B})$ by $Tx = \beta x mod 1$, and μ the T-invariant measure given by $\mu(B) = \int_B g(x)dx$, where

$$g(x) = \begin{cases} \frac{5+3\sqrt{5}}{10} & 0 \leq x < 1/\beta \\ \frac{5+\sqrt{5}}{10} & 1/\beta \leq x < 1 \end{cases}$$

b) $h_\mu(T) = -\sum_{j=1}^{m}\sum_{i=1}^{m} \pi_i p_{ij} \log p_{ij}$, where T is the ergodic markov shift on the space $(1, 2, \ldots, m^{\mathbb{Z}}, \mathcal{F}, \mu)$, with $\mathcal{F}$ is the σ-algebra generated by the cylinder sets and μ is the Markov mesure with staionary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ and transition probabilities $(p_{ij} : i, j = 1, \ldots, m)$.