Natuur- en Sterrenkunde, Universiteit Utrecht

NS-374B: Modern Astrophysics and Observational Cosmology Final Exam (2012/13)

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NB: Any cosmological data, equations or integrals required to answer the questions in this exam can be found in the attached formulas and equations.

10 Multiple Choice Questions

4% each

1. Hypernovae explosions are due to
   - (a) core collapse
   - (b) detonation of a white dwarfs at the Chandrasekhar limit
   - (c) merging of two black holes
   - (d) pair-instability
   - (e) resonances in a variable star
2. Why do we suspect that all white dwarfs observed in our galaxy were produced by the death of medium-mass stars?
   - (a) The range of white dwarf masses is narrow
   - (b) High-mass stars do not produce white dwarfs
   - (c) The age of our Galaxy is less than the life expectancy of low-mass stars
   - (d) Both (a) and (b) above
   - (e) Both (b) and (c) above
3. What unusual property do all higher-mass white dwarfs have?
   - (a) They are cooler than lower-mass white dwarfs
   - (b) They are smaller than lower-mass white dwarfs
   - (c) They are less dense than lower-mass white dwarfs
   - (d) They are less luminous than lower-mass white dwarfs
   - (e) They have a higher mass fraction X than lower-mass white dwarfs

4. Which of the following astrophysical objects is not used as a distance indicator?

   (a) Brightest red giants in a galaxy
   (b) Brightest globular clusters in a galaxy
   (c) Brightest planetary nebulae in a galaxy
   (d) Brightest active galactic nuclei in a cluster
   (e) Type Ia Supernovae

5. Which of the following quantities is not used in the description of the thermodynamic history of our Universe?

   (a) pressure
   (b) magnetisation density
   (c) entropy density
   (d) mass density
   (e) particle number density

6. In the ultra relativistic limit the number density is proportional to

   (a) T^3
   (b) T^4
   (c) $T^{3/2}$
   (d) $T^{5/2}$
   (e) $T^{3/2}e^{\frac{\mu}{k_B T}}$

7. The CMB is slightly hotter than the CνB because

   (a) photons have zero mass and neutrinos have a very small mass
   (b) neutrinos decouple from matter after photons when the Universe is colder
   (c) CMB photons are heated up by very hot gas in dense clusters of galaxies
   (d) neutrinos cool faster because they hardly interact with any matter
   (e) photons are heated up by the electron-positron annihilation

8. The heaviest nuclei produced during the primordial nucleo-synthesis is an isotope of

   (a) helium
   (b) lithium
   (c) beryllium
   (d) carbon
   (e) iron

9. Order the following events: A. quark-to-hadrons transition; B. neutrino decoupling; C. matter-radiation decoupling; D. Radiation-to-matter transition; E. Matter-to-dark energy transition

   (a) A-B-C-D-E
   (b) B-A-D-E-C
   (c) D-E-A-B-C
   (d) B-D-A-C-E
   (e) A-B-D-C-E

10. Why is the recombination rate governed by the cross section of the electron capture to the first excited state of the hydrogen atom?

   (a) statistically there are more excited states
   (b) hydrogen is move abundant than helium
   (c) an electron captured directly to the fundamental state releases a UV photon that ionises another atom
   (d) the Universe is still very dense and nearby protons exert an attraction on the recombined electron preventing it to reach the fundamental state
   (e) an electron captured directly to the fundamental state is accelerated and overshoots the nuclei

Short Questions

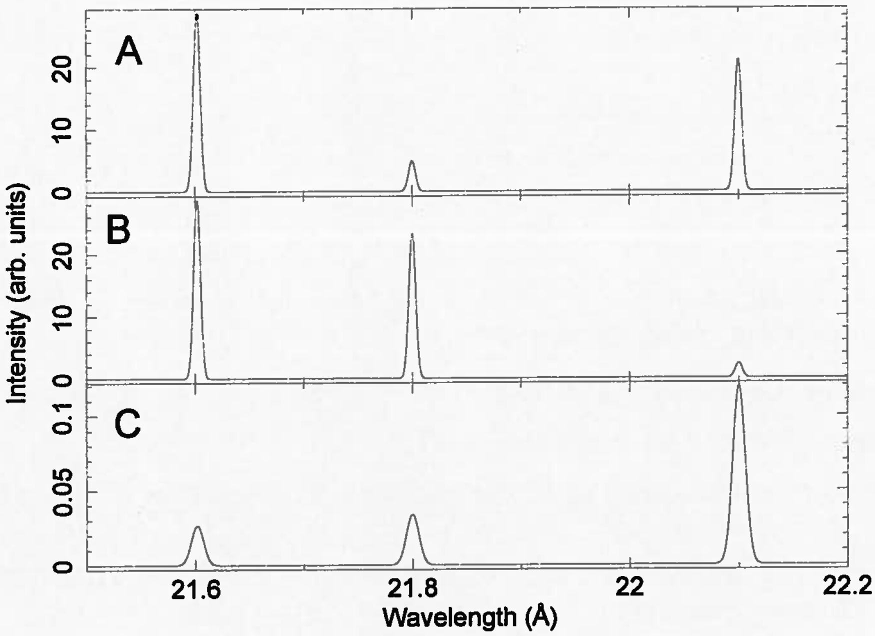
11. **Cosmology and dark matter** **8%**

   (a) Briefly describe the dynamics of the fluctuations of mass density anisotropies between the end of inflation and the matter-radiation?
   (b) How does the presence of dark matter affect the acoustic peaks of the CMB power spectrum and how does it relate to the dynamics described in (a)?

12. **Cosmic variance** **5%**

   (a) What is cosmic variance and which is the problem it poses in the analysis of CMB temperature anisotropies?
   (b) How is this problem partially solved?

13. Oxygen triplets **6%**



Oxygen triplets

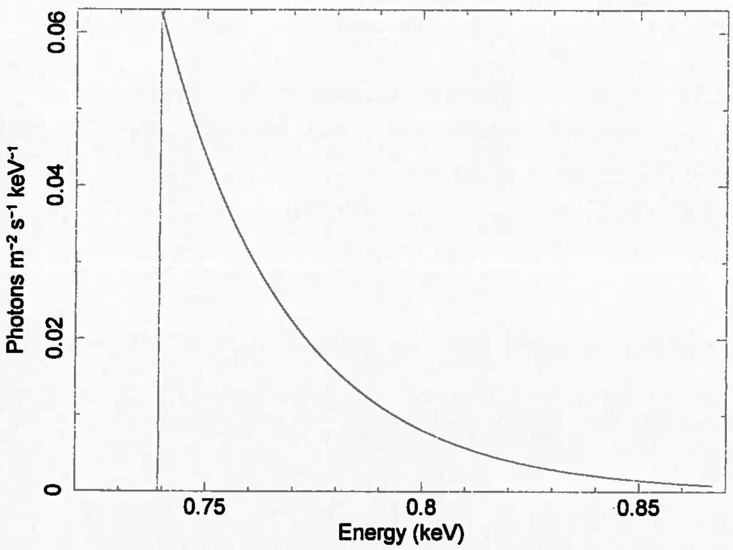
The Figure shows three X-ray spectra near the O VII triplet, labeled A, B and C. They are from three out of four different sources. In arbitrary order these sources are:

1. The hot corona of a star with a very high density
2. The hot and tenuous gas in a cluster of galaxies
3. The absorption spectrum of a Seyfert 1 galaxy
4. The photoionised emission from a Seyfert 2 galaxy

Find out to which of the four sources 1–4 each of the three spectra A, B and C belong, and explain why. Note that for one of the four sources 1–4 we do not show a spectrum.

If this is useful for you: the wavelengths of the resonance, intercombination and forbidden lines are 21.6, 21.8 and 22.2 Å.

14. **Radiative recombination continua: astrophysics using a ruler** **6%**



Radiative Recombination Continuum of O VII.

Background:
during the lectures we have discussed various ionisation equilibria. The simplest case is CIE where everything is determined by collisions between electrons and ions. Typically, you will see those ions with ionisation potential of the same order of magnitude as the temperature. In supernova remnants heating occurs rapidly and it takes some time to catch up for the ions. In photoionised plasmas the strong radiation field provides an additional source of ionisation, while it does not provide additional recombination sources, and thus it is over-ionised.

During the lectures the radiative recombination continua (RRCs) have been discussed: the spectra resulting from the capture of a free electron in a bound shell. The Figure shows the RRC from O VII, i.e. corresponding to the capture of a free electron by an O VIII ion, forming O VII during the process. The RRC emissivity dN/dE (in photons/area/time/energy) can be approximated for the circumstances of the spectrum of Figure by

$$dN/dE \sim e^{-(E-I)/kT}, \tag{1}$$

for energies $E > I$ with I the ionisation potential of O VII. It is of course zero for $E < I$. The ionisation potential of O VII is 739 eV.

Questions:

(a) Using a ruler or your trained eye, determine the temperature of the gas emitting this RRC. An accuracy within a factor of 2 is sufficient here.

(b) Given your answer to the previous question, do you think this spectrum is part of
(i) an ionising plasma like a supernova remnant,
(ii) a CIE plasma like a stellar corona or
(iii) a photoionised plasma like an AGN. Explain briefly what leads you to the identification.

Computational Questions

15. Consider the early Universe at a temperature slightly below 10^{12} K. **20%**

   (a) Which particles are still in thermal equilibrium in the cosmic bath at this temperature? Mention both relativistic and non-relativistic particles and determine the effective degeneracies g_{eff} and q_{eff}.

   (b) Electrons and positrons are kept in equilibrium mainly due to the reaction

$$e^- + e^+ \leftrightarrow 2\gamma$$

   What is the relation between the chemical potential of an electron and a positron?

   (c) Show that number density of electrons is slightly higher than the number of positrons at this stage. More specifically, show that

$$n_{e^-} - n_{e^+} \approx \frac{g}{6\hbar^3 c^2} (k_B T)^3 \frac{\mu}{k_B T} + \mathcal{O}\left((\frac{\mu}{k_B T})^3 \right)$$

   to linear order $\mu/k_B T$ which is assumed small (Justification in next question) and μ is the chemical potential of an electron. Justify the approximation you use that enable you to preform the integrals involved.

   (d) From the result in (c), which observational ratio justifies the assumption that $\mu/k_B T$ is small? Show that the assumption is consistent with this ratio.

16. Use cosmological parameter values from the Planck Mission. **15%**
Justify any approximations you use to answer the questions below.

   (a) Determine the red-shift z_c for the deceleration-acceleration transition.

   (b) Determine the red-shift z_{eq} for the matter-to-dark energy equilibrium.

   (c) Which of the two transitions in a) and b) occurred earlier in the evolution of the Universe? Show that the reverse sequence of these transitions is not possible?

Astronomical and Physical Parameters and Constants

$M_\odot = 1.989 \times 10^{30}$ kg	mass of the sun
$R_\odot = 6.9599 \times 10^{8}$ m	radius of the sun
$L_\odot = 3.826 \times 10^{26}$ J/s	luminosity of the sun
$T_\odot = 5770$ K	surface temperature of the sun
$M_\oplus = 1.989 \times 10^{30}$ kg	mass of the sun
$R_\oplus = 6.9599 \times 10^{8}$ m	radius of the sun
23 h 56 m 0.09054 s	Sideral day
86400 s	Solar day
3.155815×10^{6} s	Sideral year
3.155693×10^{6} s	Tropical year
1 ly $= 9.4605 \times 10^{15}$ m	light-year
1 pc $= 3.0857 \times 10^{16}$ m $= 3.2616$ ly	parsec
1 AU $= 1.496 \times 10^{11}$ m	Astronomical Unit
$h = 6.63 \times 10^{-34}$ J·s	Planck's constant
$e = 1.60 \times 10^{-19}$ C	elementary charge
$G = 6.67 \times 10^{-11}$ N·m^2/kg^2	gravitational constant
$c = 3.00 \times 10^{8}$ m s^{-1}	speed of light
$k_B = 1.38 \times 10^{-23}$ J/K $= 8.617$ eV/K	Boltzmann constant
$\sigma_{SB} = 5.67 \times 10^{-8}$ W·m^{-2}·K^{-4}	Stefan-Boltzmann constant
$m_H = 1.673 \times 10^{-27}$ kg	mass of the proton
$m_{He} = 6.643 \times 10^{-27}$ kg	mass of a Helium nucleus
$m_e = 9.11 \times 10^{-31}$ kg	electron mass
$r_N \approx 10^{-15}$ m $= 1$ fm (Fermi)	approximate size of atomic nucleus
$\sigma_T = 6.65 \times 10^{-29}$ m^2	Thomson cross-section
$\epsilon_0 = 8.85 \times 10^{-12}$ N·m^2/C^2	permittivity of free space

Statistical Equilibrium Equation

- Boltzmann excitation equation

$$\frac{N_i}{N_j} = \frac{g_i}{g_j}\, e^{-\frac{E_i - E_j}{k_B T}}$$

where $E_n = -\frac{m_e^4 k_B^2 Z^2}{2n^2\hbar^2}$ with $n = i, j$

- Saha Equation

$$\frac{N_{j+1}}{N_j} = \frac{2Z_{j+1}}{N_e Z_j}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\frac{\chi_j}{k_B T}} = \frac{A_j}{N_e}\,(k_B T)^{3/2}\, e^{-\frac{\chi_j}{k_B T}}$$

Expressions for spherical symmetric systems

- $$V = \frac{3}{5}\frac{GM^2}{R}$$
- $$I = \frac{1}{2}MR^2$$

Radiation

- Mass-Luminosity relation: $L \propto M^{2.7}$
- Specific intensity I_λ

$$E_\lambda d\lambda = \mathrm{I}_\lambda d\lambda\ dt\ dA\ \cos\theta\ d\Omega$$

Mean specify intensity $\bar{\mathrm{I}}_\lambda$

$$\bar{\mathrm{I}}_\lambda = \langle \mathrm{I}_\lambda \rangle = \frac{1}{4\pi}\int_\Omega \mathrm{I}_\lambda d\Omega$$

specify energy density u_λ

$$u_\lambda d\lambda = \frac{1}{c}\int_\Omega \mathrm{I}_\lambda d\lambda\ d\Omega = \frac{4\pi}{c}\bar{\mathrm{I}}_\lambda d\lambda$$

specify radiative flux F_λ

$$F_\lambda d\lambda = \int_{\Omega_{hem}} \mathrm{I}_\lambda d\lambda\ \cos\theta\ d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} \mathrm{I}_\lambda d\lambda\ \cos\theta\ d\Omega$$

radiation pressure P

$$P = \int_0^\infty P_\lambda d\lambda \quad \text{with } P_\lambda = \frac{dp_\lambda/dt}{dA}$$

and for photons with wavelength between λ and $\lambda + d\lambda$

$$P_\lambda d\lambda = \frac{2}{c}\int_{\Omega_{hem}} \mathrm{I}_\lambda d\lambda \cos^2\theta d\Omega$$

Stellar interiors

- Mean free path $\ell = \dfrac{1}{n\sigma}$

  Decrease in radiation intensity I_λ: $\quad d\mathrm{I}_\lambda = -\kappa_\lambda \rho\, \mathrm{I}_\lambda\, ds$

  Optical depth $d\tau_\lambda = -\kappa_\lambda \rho\, ds$

- Mass fractions

$$\frac{1}{\mu} = \sum_j \frac{(1+z_j)X_j}{m_j/m_H}$$

- Structure equations

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -G\frac{M(r)\rho(r)}{r^2}$$

$$P(r) = \frac{\rho(r)kT(r)}{\mu(r)m_H} + \frac{1}{3}aT(r)^4$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\, \epsilon(r)$$

$$\frac{dT}{dr} = -\frac{3\,\kappa(r)\rho(r)}{64\,\pi\sigma r^2 T^3}L(r)$$

$$\left.\frac{dT}{dr}\right|_{\text{ad}} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu(r)\, m_H\, GM(r)}{k\, r^2}$$

$$\left.\frac{dT}{dr}\right|_{\text{ad}} = \left(1-\frac{1}{\gamma}\right)\frac{T(r)}{P(r)}\frac{dP}{dr}$$

$$PV^\gamma = \text{const}$$

Stellar Remnants

- Core collapse

$$v_s > v_{\text{infall}}^{(inner)} \propto r \text{ and } v_s < v_{\text{infall}}^{(outer)} \propto 1/\sqrt{r}$$

- probability density distribution function $f(p)$

$$\text{example: } P = \frac{1}{3}\int (vp)\, 4\pi f(p) p^2 dp$$

- Pressure in a completely degenerate gas

$$P \ = \frac{8\pi}{3h^3}\int_0^{p_F} v\, p^3 dp = \ \frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}} dp$$

$$\text{with } p_F = \hbar\,(3\pi^2 n_e)^{1/3} = \hbar\,(3\pi^2 Y_e\, \rho/m_H)^{1/3}$$

- Condition for degeneracy

$$\frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3m_e k}\left(\frac{3\pi^2 Y_e}{m_H}\right)^{2/3}$$

- Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\,\theta^n$$

- Chandrasekhar Mass

$$M_{\text{Ch}} \approx \frac{\sqrt{6}}{32\pi}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{Y_e}{0.5}\right)^2 \frac{2.018}{m_H^2} \approx 1.457\left(\frac{Y_e}{0.5}\right)^2 M_\odot$$

- Conservative lower bound for the stability of a rotating dense object

$$\rho_{lb} > \frac{3\pi}{GP^2}$$

- Pulsar period increase rate

$$2\pi\frac{\dot{P}}{P^2} I = \frac{2\pi}{3}\,\frac{B_P^2 R^6 \sin^2\theta}{\mu_0^2 c^3}\,\frac{8\pi^3}{P^3}$$

Cosmological Parameters

$H_0 = 70.4^{+1.3}_{-1.4}$ km s^{-1} Mpc^{-1} (often used 70.7)	Hubble constant (WMAP)
$h: \; H_0 = 100\,h$ km s^{-1} Mpc^{-1}	reduced Hubble constant
q_0	deceleration parameter
$\rho_c = \dfrac{3H^2}{8\pi G}$	critical density
$\rho_{c,0} = \dfrac{3H_0^2}{8\pi G} \approx 1.88 \times 10^{-26} h^2$ kg/m^3	critical density at present
$\Omega = \rho/\rho_{\text{crit}}$	density parameter
$\Omega_0 = \dfrac{\rho_0}{\rho_{c,0}} = 1 + \dfrac{\kappa c^2}{H_0^2 a^2}$	density parameter at present
Ω_B	Baryonic matter density parameter
$n_\gamma \approx 2.67 \times 10^{-8}\,(\Omega_{\text{B},0}\,h^2)$ $\eta_\gamma = \dfrac{n_\text{B}}{n_\gamma} \approx 2.67 \times 10^{-8}\,(\Omega_{\text{B},0}\,h^2)$	Baryon-to-photon ratio
$\eta_s = \dfrac{n_\text{B}}{s} \approx \dfrac{\eta_\gamma}{7.04}$	Baryon-to-entropy ratio

Planck Mission best fit data (March 2013)

$H_0 = 67.8 \pm 0.77$ km s^{-1} Mpc^{-1}	Hubble constant
$h = 0.678$	reduced Hubble constant
$t_0 = 13.798 \pm 0.037$ billion years	Age of the Universe
$\Omega_{\text{B},0} = 0.048999$	Baryonic matter density parameter
$\Omega_{\text{DM},0} = 0.26709$	Dark matter density parameter
$\Omega_{\Lambda,0} = 0.6825$	Dark energy density parameter
$n_s = 0.9624$	Spectral index

Friedmann Models

- Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{a^2}$$

- Conservation of Energy Equation

$$\dot{\rho} = -3\,\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)$$

- Acceleration Equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right)$$

- Equations of State

$$p(\rho) = \omega\rho, \;\; \omega \text{ depends on the medium}$$

$$\rho(a) = \rho_0 \left(\frac{a_0}{a}\right)^{3(\omega+1)}$$

$$a(t) \propto t^{\frac{2}{3(\omega+1)}} \;\; \text{for } \omega \neq -1 \;\; \text{and} \;\; a(t) \propto \mathrm{e}^{Ht} \;\; \text{for } \omega = -1$$

- Cosmological red-shift

$$z = \frac{a_0}{a} - 1$$

- Negative curvature dominated universe

$$a(t) \propto t$$

Thermodynamics of the Early Universe

- Number density

$$n = \frac{g}{2\pi^2\hbar^3c^2}\int_{mc^2}^{\infty}\frac{E\sqrt{E^2/c^2-m^2c^2}}{e^{\beta(E-\mu)}\pm 1}\,dE$$

$$\text{in URL:}\quad n = \begin{cases} g\dfrac{\zeta(3)}{\pi^2}\left(\dfrac{k_BT}{\hbar c}\right)^3 & \text{for bosons} \\ \dfrac{3}{4}g\dfrac{\zeta(3)}{\pi^2}\left(\dfrac{k_BT}{\hbar c}\right)^3 & \text{for fermions}\end{cases} \qquad \text{with}\quad \zeta(3)\approx 1.2020569$$

$$\text{in NRL:}\quad n = g\,e^{-\frac{mc^2-\mu}{k_BT}}\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$$

- Mass density

$$\rho = \frac{g}{2\pi^2\hbar^3c^4}\int_{mc^2}^{\infty}\frac{E^2\sqrt{E^2/c^2-m^2c^2}}{e^{\beta(E-\mu)}\pm 1}\,dE$$

$$\text{in URL:}\quad \rho = \begin{cases} \dfrac{g\pi^2k_B^4}{30\hbar^3c^5}T^4 & \text{for bosons} \\ \dfrac{7}{8}\dfrac{g\pi^2k_B^4}{30\hbar^3c^5}T^4 & \text{for fermions}\end{cases}$$

$$\text{in NRL:}\quad \rho = n\,m$$

- Pressure

$$p = \frac{g}{6\pi^2\hbar^3}\int_{mc^2}^{\infty}\frac{(E^2/c^2-m^2c^2)^{3/2}}{e^{\beta(E-\mu)}\pm 1}\,dE$$

$$\text{in URL:}\quad p = \frac{1}{3}\rho c^2$$

$$\text{in NRL:}\quad \rho = n\,k_BT$$

- Entropy density in non-degenerate regime

$$s \equiv \frac{1}{T}(\rho c^2 + p)$$

- Useful integrals

$$\int_0^{\infty}\frac{x^3\,dx}{e^x\pm 1} = \begin{cases} \dfrac{\pi^4}{15} & \text{for bosons} \\ \dfrac{7}{8}\dfrac{\pi^4}{15} & \text{for fermions}\end{cases}$$

$$\int_0^{\infty}\frac{x^2\,dx}{e^x\pm 1} = \begin{cases} 2\,\zeta(3) & \text{for bosons} \\ \dfrac{3}{2}\zeta(3) & \text{for fermions}\end{cases}$$

$$\int_0^{\infty}\frac{x^2\,e^x dx}{(e^x\pm 1)^2} = \begin{cases} \dfrac{\pi^2}{3} & \text{for bosons} \\ \dfrac{\pi^2}{6} & \text{for fermions}\end{cases}$$

- Effective degeneracies

$$g_{\text{eff}} \equiv \sum_i^{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_j^{\text{fermions}} g_j \left(\frac{T_j}{T}\right)^4$$

$$q_{\text{eff}} \equiv \sum_i^{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_j^{\text{fermions}} g_j \left(\frac{T_j}{T}\right)^3$$

Temperature of important Epochs

- Time-Temperature relation in RDU

$$t \approx \frac{0.301\, \hbar}{\sqrt{g_{\text{eff}}(T)}} \frac{m_{\text{Pl}} c^2}{(k_B T)^2} \quad \text{with} \quad m_{\text{Pl}} = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19}\ \text{GeV/c}^2$$

- Electroweak transition: $T_{\text{EW}} \approx 10^{14}$ K $(k_B T \approx 200$ GeV)
- Quark-Hadron transition: $T_{\text{q-h}} \approx 10^{12}$ K $(k_B T \approx 1$ GeV)
- Neutrino decoupling and electron-positron annihilation: $T_{\text{D}} \approx 10^{10}$ K $(k_B T \approx 1$ MeV)
- Threshold for neutron decay: $k_B T \approx 0.1$ MeV
- Deuterium production: $k_B T \approx 0.07$ MeV $< B_D \approx 2.2$ MeV
- Radiation-to-matter transition: $T_{\text{eq}} \approx 14500$ K
- Recombination: $T \approx 3000$ K $(k_B T \approx 0.308$ eV)