OBSERVATIONAL & THEORETICAL COSMOLOGY

Midterm Exam 25.05.2016

■ PROBLEM 1 (3 points) : Theoretical questions

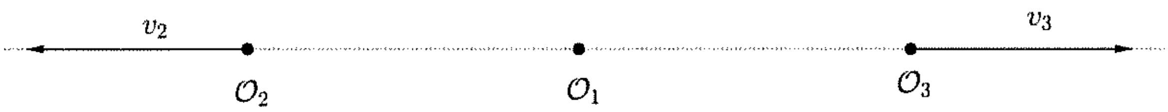

Figure 1: The geometry of the three observers from theoretical Question 1.

Question 1: Consider the configuration of three observers from Figure 1. The observer $\mathcal{O}_1$ is standing still, while $\mathcal{O}_2$ and $\mathcal{O}_3$ move away from $\mathcal{O}_1$ in opposite directions. In $\mathcal{O}_1$ frame, $\mathcal{O}_2$ and $\mathcal{O}_3$ are moving with respect to each others with speed $v = v_2 + v_3$. Clearly there are choices of v_2, v_3 such that $v > c$ (for example $v_2 = 0.6c$ and $v_3 = 0.8c$). Why this is not in contradiction with Lorentz invariance?

Question 2: What is the weak equivalence principle and why does it imply that the gravitational mass is equal to the inertial mass?

Question 3: The graviton is the propagating degree(s) of freedom of gravity, and describes perturbations of the metric around a flat background, *i.e.*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \, .$$

How many degrees of freedom does the graviton $h_{\mu\nu}$ carry in a three dimensional space-time (two spatial dimensions and one time dimension)? Justify your answer.

■ PROBLEM 2 (3 points): Uniformly accelerating observer

Let $\mathcal{O}_1$ be an observer at rest, and $\mathcal{O}_2$ an uniformly accelerated observer. The trajectory of $\mathcal{O}_2$ in $\mathcal{O}_1$ frame is described by the parametric curve (in this problem we set $c = 1$ such that time is measured in meters):

$$\begin{pmatrix} t(\lambda) \\ x(\lambda) \\ y(\lambda) \\ z(\lambda) \end{pmatrix} = \begin{pmatrix} a\sinh/(\lambda/a) \\ a\cosh/(\lambda/a) \\ 0 \\ 0 \end{pmatrix} . \tag{2.1}$$

(a) Show that λ is the proper time along the world line, and give the interpretation of the parameter a.

(b) Draw a space-time diagram (in the (x, t) plane) in which you show:

(i) The trajectory of $\mathcal{O}_2$, that is the curve,

$$\begin{pmatrix} t(\lambda) \\ x(\lambda) \end{pmatrix} .$$

(ii) The space-time region that can send light signals to $\mathcal{O}_2$.

(iii) The space-time region that can receive signals from $\mathcal{O}_2$.

(c) Define the Momentarily Comoving Reference Frame and compute the proper acceleration of the observer $\mathcal{O}_2$ and show that the components of the proper acceleration are,

$$a^\mu = \begin{pmatrix} 0 \\ a^x \\ 0 \\ 0 \end{pmatrix}, \qquad \text{with} \quad a^x = 1/a.$$

■ PROBLEM 3 (4 points) : Detection of gravitational waves

Consider a gravitational wave $h_{ij} = h_{ij}(t,y)$ propagating in the positive y-direction ($\vec{k} = k\ e_{(y)}$). There are two independent polarisations of this gravitational wave, a *plus* one and a *cross* one. In the case at hand, one can write $h_{ij}(t,y) = \sum_{a=(+,\times)} \epsilon^a_{ij} h^a(t,y)$, where ϵ^+_{ij} and $\epsilon^\times_{ij}$ are the following polarization tensors[1]

$$\epsilon^+_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \epsilon^\times_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

and

$$h^a(t,z) = A^a \cos(\omega t - ky). \tag{3.2}$$

We want to determine what is the effect this wave has on a group of test particles initially distributed in a circle perpendicular to the propagation direction of the wave, *i.e.* the circle lies in the (xz)-plane. In order to do that one needs to solve the geodesic deviation equation for the separation vector S^σ between two nearby particle trajectories, which in leading order for slow moving particles ($dx^\alpha/d\tau \approx (1,0,0,0)$) is given by

$$\frac{\partial^2}{\partial t^2} S^i = \frac{1}{2} S^j \frac{\partial^2}{\partial t^2} h_{ij}. \tag{3.3}$$

(a) Consider a gravitational wave as in (3.2) with $A^\times = 0$. Show that (3.3) has the following perturbative solution (A^+ serves as a perturbation parameter in equation (3.3))

$$S^1(t,y) = \left(1 + \frac{A^+}{2}\cos(\omega t - ky)\right) S^1(0,y), \tag{3.4}$$

$$S^2(t,y) = S^2(0,y), \tag{3.5}$$

$$S^3(t,y) = \left(1 - \frac{A^+}{2}\cos(\omega t - ky)\right) S^3(0,y). \tag{3.6}$$

(b) Derive the solution of (3.3) for a gravitational wave for which $A^+ = 0$, i.e. when the wave is cross-polarised.

(c) Sketch for both (plus and cross) polarisations of the above gravitational wave how particles move assuming they were distributed on a circle at the initial moment. Do this by sketching snapshots of the particle distribution for every 1/4 of the period.

[1] In this problem the normalisation factor of $\frac{1}{\sqrt{2}}$ can be ignored.