Exam for Part II of NS-374B
Observationele en theoretische kosmologie

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June 21, 2017

About this exam

Exam All problems are worth one point. *You can skip at most 4 problems.* The best 14 problems will be evaluated, so your *maximum grade is 14 points.*

Strategy Roughly speaking it should take you about 10 minutes to solve every problem. Many problems can actually be solved much more quickly. Don't waste too much time in trying to re-derive the answers to a very unfamiliar question. If you blank out on a question move on to the next one. You are allowed to skip two short problems without a repercussion on your final grade.

Answers In answering the questions, be brief and concise, but at the same time specific and precise. If you do not know the answer, do not copy and paste from your notes a random fact. When a numerical answer is required, allow yourself to approximate the calculation to a single significant digit. The right order of magnitude is better than three digits of the wrong number.

Timing You have 3 hours. As a guideline, you might want to try to dedicate in average 10 minutes to each problem.

Allowed material You are allowed to bring on A4 sheet of paper with whatever you want on it. You may use electronic devices only if you use them to access this material or perform numerical calculations, but not to browse the internet or contact someone.

Topic The midterm exam will cover all the material of part II of the course, in particular all chapters of Ryden's book except chapter 10 on nucleosynthesis.

Problems (1 point each, you can skip 4 problems)

1. (2016, 2017) Give at least one example of a physical system (an actual system, not arrows drawn on paper) that is i) homogeneous but not isotropic [1/3 point], ii) isotropic but not homogeneous [1/3 point] and iii) homogeneous and isotropic [1/3 point].

2. (2016, 2017) List the known constituents of the universe. For each constituent [1/4 point], specify: the current fraction of the total energy density, how it is measured and how its energy density ϵ scales with the expansion of the universe. [Hint: be brief. At most two lines per constituent.]

3. (2017) Since you are made mostly of water, you are very efficient at absorbing microwave photons. If you were in intergalactic space, approximately how many CMB photons would you absorb per second [1/2 point]? (If you like, you may assume you are spherical.) What is the approximate rate, in watts, at which you would absorb radiative energy from the CMB [1/2 point]? [Hint: the CMB temperature today is approximately 3 K]

4. (2017) Write down the spatially flat FLRW spacetime metric. What are its isometries? [1/2 point] Now change coordinates to a boosted observer moving with constant velocity v in the $\hat{x}$ direction

$$t' = \gamma(t + vx)\,, \quad x' = \gamma(x + vt)\,, \quad y' = y\,, \quad z' = z\,. \tag{1}$$

   Derive the FLRW metric in the new coordinates [1/2 point]. What are its isometries now? [bonus 1/2 point][Hint: to invert the Lorentz transformation you might simply switch primed coordinates with unprimed and $v \to -v$.]

5. Derive the fluid (a.k.a. continuity) equation

$$\dot{\epsilon} + 3H(\epsilon + p) = 0\,. \tag{2}$$

from the first law of thermodynamics. [Hint: recall that the total energy is $E = \epsilon V$, and the expansion is adiabatic]

6. Consider a de Sitter universe, namely a universe with only a cosmological constant. Assume it is spatially flat. Write down and solve the Friedmann equation to find $a(t)$ explicitly. Write down and solve the continuity equation to find $\epsilon_\Lambda(t)$. In what stages of the history of our universe, was our spacetime well approximated by this solution?

7. Write down and *study* the Friedmann equation, including arbitrary spacetime curvature [Hint: see Appendix on how to study an equation. Keep it brief. No more than 2 lines per step, total of max 10 lines].

8. Derive an expression for the scale factor $a_{m\Lambda} = a(t_{m\Lambda})$ at matter-Lambda equality, namely when $\epsilon_m(a_{m\Lambda}) = \epsilon_\Lambda(a_{m\Lambda})$, in terms of today's fractional matter and Lambda energy densities $\Omega_{\Lambda,0}$ and $\Omega_{m,0}$.

9. What would be the age of our universe if there were only radiation, namely $\Omega_{r,0} = 1$, assuming $H_0^{-1} \simeq 2 \times 10^{17}$ sec? Derive the solution by first proving that $dt = da/(aH)$ and then integrating this expression and using the algebraic solution $H(a)$ of Friedmann equation in the presence of just radiation (zero spatial curvature).

10. In a single component universe, with some equation of state parameter w, derive the relation between the current proper distance

$$d_p(t) \equiv c \int_t^{t_0} \frac{dt'}{a(t')} \tag{3}$$

and the redshift $z(t)$. When does is this relation $d(z)$ linear? [Hint: this requires two distinct answers. First, $d(z)$ is always linear for some specific w. Second, $d(z)$ is approximately linear for every w in a specific regime]

11. In a spatially flat universe containing a single perfect fluid, derive the luminosity distance d_L and the angular-diameter distance d_A as a function of redshift z and equation of state parameter w [1/2 point]? Assuming $w = 0$, at what redshift will d_A have a maximum value [1/2 point]?

12. (2017) Assume that the Hubble radius is known, $1/H_0 = 4$ Gpc, and that you are given the following data for a supernova Ia

$$z = 0.1\,, \quad m - M = 38.2\,. \tag{4}$$

with negligible error bars. Is the expansion of the universe accelerated or decelerated? [Hint: Since we have only low redshift data $z \ll 1$, we can use the Taylor expansion of the scale factor

$$a(t) = 1 + H_0\,(t - t_0) - \frac{q_0}{2} H_0^2\,(t - t_0)^2\,. \tag{5}$$

Recall that apparent magnitude m, absolute magnitude M and luminosity distance are related by

$$m - M = 5 \log_{10}\left(\frac{d_L}{\text{Mpc}}\right) + 25\,. \tag{6}$$

Also, the luminosity distance d_L is related to the proper distance d_p, which in the Taylor expansion above is given by

$$d_p = \frac{z}{H_0}\left(1 - \frac{1 + q_0}{2} z\right)\,. \tag{7}$$

Finally, keep in mind that a dimensionless quantity such as q_0 can be computed in any units you like. You should not need to specify the value of c.]

13. Consider a star following a circular orbit around a spherically symmetric galaxy with radial mass density

$$\rho(R) = \bar{\rho}\,\exp\left(-\frac{R}{R_s}\right)\,, \tag{8}$$

with $R_s = 5$ kpc. Derive a formula for the dependence of the velocity of the star on the distance R from the center of the galaxy. How does this result compare with the orbital velocity of real, observed stars?

14. Photons are electro-magnetically neutral particles and therefore do not interact with each other (at least classically). Yet the spectrum of the CMB is a perfect black body, which is a solution of thermodynamic equilibrium. What interaction in the early universe allowed the photons to remain in thermodynamic equilibrium? When did this interaction cease to be effective?

15. Explain what recombination is (max two lines). Why is the temperature at recombination about 0.3 eV, which is 40 times smaller than the binding energy of the Hydrogen atom?

16. [Hint: some might find this question a bit harder] Starting from the Saha equation (in units of the Boltzmann constant $k_B = 1$)

$$\frac{n_H}{n_p n_e} = \left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{Q/T}, \tag{9}$$

where Q is the binding energy of the Hydrogen atom, $n_{e,p}$ the number density of free protons and electrons and n_H that of neutral Hydrogen, derive the dependence of the recombination temperature T_{rec} at which $n_e = n_H$ on the proton-to-photon ratio $\eta \equiv (n_p + n_H)/n_\gamma \sim 10^{-10}$ [1/2 point]. In particular, estimate the change in T_{rec} if η is twice as large [1/2 point]. [Hint: you may forget about the neutrons and assume all baryons in the universe are protons. You can also use charge neutrality of the universe $n_e = n_p$. Keep in mind the approximate reference value $T_{rec} \sim 0.3$ eV. Finally, since the transcendental equation cannot be solves, you should focus on the T_{rec} dependence in the in the exponential.]

17. State the horizon problem (max two lines). How does inflation address this problem?

18. Derive the equations of motion for the homogenous evolution of the inflaton field $\phi(t)$ from the action

$$S = \int dt\, a^3 \left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right], \tag{10}$$

where $a = a(t)$ is the scale factor. [Hint: impose $\delta S = 0$ for arbitrary $\delta\phi$]

Appendix: How to study an equation

To *study* an equation you should:

1. **Form** Stare at the equation as you would stare at a beautiful painting. Take at least 30 seconds to just look at it. Discover all of its tiny indices, hidden dependences, overall form. Is it an algebraic or differential equation? If differential, to what order? Is it partial or ordinary?

2. **Variables** Enumerate and characterize the variables in the equations: what are they functions of, how do they appear (e.g. with or without derivatives, integrated over, implicitly, ...).

3. **Dimensional analysis** Know/review the mass dimension (and/or other dimensions if $\hbar \neq 1 \neq c$) of every single parameter, variable and function appearing in the equation. Be sure to master this. Dimensional analysis is the grammar of physics.

4. **Symmetries** Discuss the symmetries of the equation: is it covariant (i.e. invariant in form) under change of coordinates/rotations/translations? is it exactly/approximately invariant under some other symmetry? How do you build new solutions from known ones?

5. **Limits** Enumerate simple limits in which the equation takes a simple, well-known or intuitive form or in which you know a (simple) solution.