Exam Fluid Mechanics and Transport Phenomena 12 April 2017, 13:30-16:30

Answers may be given in English or Dutch. Please answer exercises on separate sheets, always listing name, student nr, and question nr. Each subquestion is 1 point and gives the relative weight (total amount of points equals a mark 10).

1 Consider the motion of a single colloid of mass m in a solvent. Assume that the colloid experiences a drag force that is well described by Stokes' Law where the friction coefficient is given by f_{tr}. For this problem, we will only consider the x-component of the motion. Assume that the colloid is undergoing Brownian motion, and that the associated random force in the x-direction is given by $R(t)$.

- (a) What are the properties of the random force $R(t)$?
- (b) Write down the Langevin equation of motion for the x-component.
- (c) Assume that the system is overdamped, that is to say, that the inertial forces are much smaller than the viscous forces. Argue that in this limit, the equation of motion becomes a first order stochastic differential equation. Write down this first order stochastic differential equation. Show that this differential equation has the solution
$$x(t) = \frac{1}{f_{tr}} \int_0^t R(t')dt' \tag{1}$$
where we assume that $x(0) = 0$.
- (d) Determine $\langle x(t) \rangle$ for this system. Is this what you would expect? Explain.
- (e) Determine $\langle x^2(t) \rangle$ for this system. Is this what you would expect? Explain.

2 Answer sub-questions first of all with a simple 'yes' or 'no' and then explain your answer.

- (a) A two dimensional flow in the x, y-plane is given by $(u, v) = (x^2 - y, -2xy)$. Is it true that this velocity field can be derived from a potential?
- (b) Is it true that the acceleration of a fluid parcel in point $(x, y) = (2, 4)$ at time $t = 1$, which is moving with the flow given in (a), equals $(16, 32)$?
- (c) Is it true that knowledge of the stream function field is relevant for determining the vector velocity field?
- (d) Is it sufficient for the appearance of boundary layer separation, that a boundary layer's velocity profile, $u(y)$, requires the presence of an inflexion point within the boundary layer (i.e. a point where $\partial^2 u/\partial y^2 = 0$)? Here x, y are boundary layer coordinates, parallel and perpendicular to the boundary respectively, and u is the velocity in x-direction.

- (e) Is it true that

$$\epsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} = 0?$$

Here x_i refer to coordinates x_1, x_2, x_3, and ϵ_{ijk} denotes the antisymmetric tensor.

- (f) Is it true that an object can acquire lift only when it is shaped asymmetrically?

- (g) Is the acceleration of a fluid parcel, at time t situated at $\mathbf{r}(t, \mathbf{r_0})$, the same as the acceleration field at that position and time?

- (h) Consider a uniform parallel flow of magnitude U around a rotating cylinder of radius a. Owing to its rotation, the cylinder produces a circulation Γ. The resulting stream function is given by

$$\psi = U \sin\phi \left(r - \frac{a^2}{r}\right) - \frac{\Gamma}{2\pi}\ln\frac{r}{a} + const.$$

Is it true that for $\Gamma \gg 4\pi a U$ the stagnation point is at $(x, y) = (0, 2a)$?

3 Shallow water waves

- (a) Consider a constant-density fluid. By choosing appropriate scales for the dependent and independent variables, derive the dimensionless version of the Navier-Stokes equations in Cartesian coordinates:

$$\mathrm{St}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - \frac{1}{\mathrm{Fr}}\hat{\mathbf{z}} + \frac{1}{\mathrm{Re}}\nabla^2\mathbf{u},$$

where $\hat{\mathbf{z}}$ denotes a unit vector pointing upwards (anti-parallel to gravity). What do the resulting dimensionless parameters mean?

- (b) In what limit are *both* the viscous as well as the nonlinear terms small? Show that in that case, in a two-dimensional fluid one obtains the following momentum equations:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

- (c) We have a fluid basin whose size L in horizontal x-direction, is much larger ($L \gg H$) than its mean depth in vertical z-direction, H. What does this imply for the pressure? Hint: scale the incompressibility equation, $\nabla\cdot\mathbf{u} = 0$ using different scales in x and z directions.

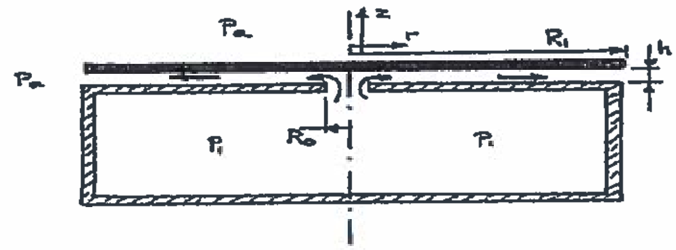
- (d) Assume the bottom is flat, the side walls vertical, and the water surface in free contact with the atmosphere. What boundary conditions apply on (i) the solid boundaries and (ii) the free surface, $H + \zeta(x, t)$, where perturbation surface level $\zeta(x, t) \ll H$? Discuss how the exact boundary conditions can be linearised.

- (e) Show that the horizontal pressure gradient can be expressed in terms of the gradient in the free surface: $\partial p/\partial x = \rho g \partial \zeta/\partial x$. What does this imply for the vertical dependence of the *horizontal velocity*?

- (f) Integrate the incompressibility equation over the vertical and show that the free surface perturbation, $\zeta(x,t)$ is governed by the following wave equation:

$$\frac{\partial^2 \zeta}{\partial t^2} - gH\frac{\partial^2 \zeta}{\partial x^2} = 0.$$

- (g) Derive a time-periodic solution of this equation that satisfies all boundary conditions.



4 A cylindrical vessel contains a viscous, incompressible fluid at constant pressure p_1. Suppose liquid flows out radially through a circular outlet of diameter $2R_0$ through a very narrow slit (of width $h \ll R_0$) between the upper side of the vessel and a massive cover plate (see figure). The pressure of the surroundings is p_a. The radius of the cover plate, R_1 is much larger than the slit width h. One may assume that the flow in the slit is stationary and axisymmetric. The kinematic viscosity ν of the liquid is very high, so that inertia forces can be neglected in the flow through the slit, compared to viscous forces. Gravity plays no role in this set-up. For the description of the flow we use a cylindrical coordinate system r, ϕ, z.

- (a) Using mass conservation, show that the radial velocity component of the flow through the slit, u_r, can be written as $u_r = c(z)/r$, where $c(z)$ is a function of axial coordinate z.
- (b) Argue that the radial component of the Navier-Stokes equations for this flow simplifies to

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \nu \frac{\partial^2 u_r}{\partial z^2}.$$

- (c) Show that the pressure in the slit is independent of z.
- (d) Give the boundary conditions on u_r and determine $u_r(r, z)$ explicitly in terms of r and z.
- (e) Give the boundary conditions for the pressure and determine its variation $p(r)$.

One then fills the vessel with a fluid having a *very small* viscosity, so that in the flow through the slit viscous forces are now negligible compared to stationary inertia forces. In the slit a radial flow will establish itself having a radial velocity $u_r = a/r$, where a is constant.

- (f) Explain why. Using the Navier-Stokes radial momentum equation, show that also in this case a relatively simple force balance exists and that Bernoulli's law applies.

- (g) For this inviscid flow, determine pressure distribution $p(r)$ assuming that $R_0 \ll R_1$ and that the pressure reduces to the surrounding pressure far away from the cover and vessel.
- (h) Determine the overall force on the cover produced by the pressure distribution.