Exam "Wave Attractors"

27 June 2018, 13:30-16:30

No books or lecture notes allowed. **Computations can all use rounded estimates**. *Weight of question is indicated in points (pt) - 34 points in total.*

Loch Ness

Loch Ness is a fresh water lake in Scotland, centered at $57^o 2' N$ and $4^o 35' W$. The lake is approximately 37 km long, 2 km wide and 200 m deep. It is a geological fault, filled in with sediments giving the Loch a nearly symmetric, trapezoidal cross-sectional shape (see figure). Its sidewalls both slope over a distance of approximately 0.5 km. For our purpose the Loch may serve as a "mini-ocean".

In summer, the water of the lake is linearly stratified in temperature, varying from 5^o C at the bottom (density $\rho_b \approx 1000\ kg\ m^{-3}$), to 16^o C at the top ($\rho_t \approx 999\ kg\ m^{-3}$).

1 (2pt) Compute an approximate value for the stability (buoyancy) frequency, N, and assume it to be constant throughout the basin.

2 (1pt) Knowing that the speed of sound in water c_s is approximately $1500 m/s$, determine whether it is relevant to take compressibility of water for baroclinic (internal wave) motions into account?

3 (4pt) In a Cartesian x, y, z frame of reference with velocity vector u, v, w, perturbation pressure $p = (p_* - p_0)/\rho_*$ and perturbation buoyancy $b = -g\rho'/\rho_*$, related to perturbation density ρ' scaled with uniform reference density ρ_*, the linearized, inviscid equations governing

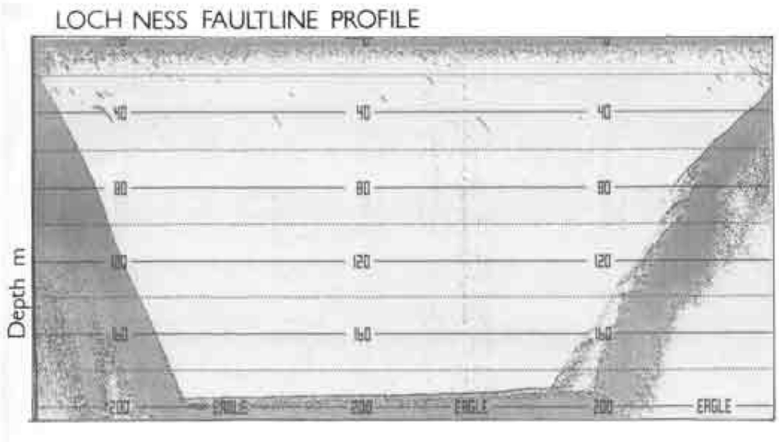

Figure 1: Transverse profile of the bathymetry of Loch Ness. At the surface, the Loch is 2 km wide. Model the sloping side walls as being 0.5 km wide each.

perturbations of this uniformly-stratified fluid are, in Boussinesq-approximation, given by:

$$\begin{aligned} u_t &= -p_x \\ v_t &= -p_y \\ w_t &= -p_z + b \\ b_t + wN^2 &= 0 \\ u_x + v_y + w_z &= 0. \end{aligned}$$

Here subscript-derivative notation is used, z points antiparallel to gravity and x and y point in the along and across Loch directions respectively.

Derive the equation that governs the spatial structure for free internal waves propagating strictly in the transverse (y, z) plane.

4 (1pt) Consider monochromatic, plane waves and show that the buoyancy frequency acts as a high-frequency cut-off.

5 (1pt) What happens to perturbations of frequency higher than this cut-off?

6 (2pt) What boundary conditions do these waves need to satisfy on the rigid sloping walls, bottom and free surface?

7 (1pt) Discuss the scaling by means of which the spatial structure of monochromatic internal waves propagating in cross-Loch direction is governed by the dimensionless equation

$$\psi_{yy} - \psi_{zz} = 0,$$

in terms of the spatial part of the stream function field $\psi(y, z)$.

8 (1pt) In terms of these scaled variables, determine the location of the boundaries and boundary conditions.

9 (1pt) Discuss why internal gravity waves that propagate in transverse y-direction would be focused onto wave attractors.

10 (2pt) Compute the range of frequencies over which 'simple' wave attractors exist, that is, having two surface reflections and one reflection from each side wall (use graphs to support your argument). (You get 0.5pt if you compute one frequency that falls within this range).

11 (1pt) Are there any transversely propagating internal waves that do *not* focus onto an attractor?

12 (1 pt) For internal waves that also propagate down-channel, give a qualitative argument why internal gravity waves might still be trapped.

13 (2pt) In *fall*, due to wind mixing and sheared bottom currents the water is well-mixed near the surface and near the bottom, in the upper and lowest 75 m of the Loch. The temperature jump between the cold bottom and warm surface layer is thus restricted to the middle 50 m. Discuss qualitatively in what way this affects the internal gravity waves within the whole water column?

14 (2pt) In *winter*, the entire water column is well-mixed, so that the Loch is homogeneous in density. Now consider that the Loch is on a rotating planet, rotating at angular velocity $\Omega = 7.2 \times 10^{-5} rad/s$. Why does the Loch still support *internal* waves, and what kind of waves?

15 (4pt) In traditional approximation, the equations governing these perturbations (waves)

are now given by

$$
\begin{aligned}
u_t - fv &= -p_x \\
v_t + fu &= -p_y \\
w_t &= -p_z \\
u_x + v_y + w_z &= 0,
\end{aligned}
$$

where Coriolis frequency $f = 2\Omega \sin\phi$.

Taking into account the small aspect ratio of the Loch (depth 200 m, width 2 km and length 37 km), use scaling to write down approximate equations governing linear *surface* wave motions in this homogeneous-density fluid?

16 (1pt) On what grounds can rotational effects on *surface* gravity waves be neglected?

17 (1pt) Now further idealize the cross-sectional shape of the Loch to a rectangle (of width 1.5 km) and assume the Loch to be infinitely long. Compute *internal* wave solutions for this homogeneous-density rotating fluid, assuming these waves propagate in along-Loch x-direction.

18 (2pt) What are the Loch's transverse eigenfrequencies, if any? Can there be any attractors in this case? Why, or why not?

19 (2pt) What is the difference between *inertial oscillations* and *inertial waves*?

20 (2pt) Now assume stratification and rotation are both present. In a uniformly-stratified ($N = const$), rotating fluid, in which rotation axis $\mathbf{\Omega}$ is anti-parallel to the direction of gravity $\mathbf{g}$, plane monochromatic internal waves of frequency ω satisfy a dispersion relation given by

$$
\omega^2 = N^2 \cos^2\alpha + f^2 \sin^2\alpha = \frac{N^2 k^2 + f^2 m^2}{k^2 + m^2},
$$

where 2D wave vector $\mathbf{k} = (k, m) = \kappa(\cos\alpha, \sin\alpha)$.

Compute the phase and group velocity vectors of these waves and show that they are orthogonal to one another?