# Exam "Wave Attractors"

27 June 2018, 13:30-16:30

*No books or lecture notes allowed.* **Computations can all use rounded estimates**. *Weight of question is indicated in points (pt) - 34 points in total.*

**Loch Ness**

Loch Ness is a fresh water lake in Scotland, centered at $57^o2'N$ and $4^o35'W$. The lake is approximately 37 km long, 2 km wide and 200 m deep. It is a geological fault, filled in with sediments giving the Loch a nearly symmetric, trapezoidal cross-sectional shape (see figure). Its sidewalls both slope over a distance of approximately 0.5 km. For our purpose the Loch may serve as a "mini-ocean".

In summer, the water of the lake is linearly stratified in temperature, varying from $5^o$ C at the bottom (density $\rho_b \approx 1000\ kg\ m^{-3}$), to $16^o$ C at the top ($\rho_t \approx 999\ kg\ m^{-3}$).

1 (2pt) Compute an approximate value for the stability (buoyancy) frequency, $N$, and assume it to be constant throughout the basin.

A finite difference approximation based on the bottom and surface density, using the classical definition of stability frequency, reads

$$N \approx \left(-\frac{g}{\rho_*}\frac{d\rho_0}{dz}\right)^{1/2} \approx \left(\frac{10\times 1}{1000\times 200}\right)^{1/2} \text{rad } s^{-1} \approx (50\times 10^{-6})^{1/2}\text{rad } s^{-1} \approx 7\times 10^{-3}\text{rad } s^{-1}$$

2 (1pt) Knowing that the speed of sound $c_s$ is approximately $1500 m/s$, determine whether it is relevant to take compressibility of water into account?

This classical definition has to be corrected for compressibility effects. Using a more correct definition of

$$N^2 = -\frac{g}{\rho_*}\frac{d\rho_0}{dz} - \left(\frac{g}{c_s}\right)^2,$$

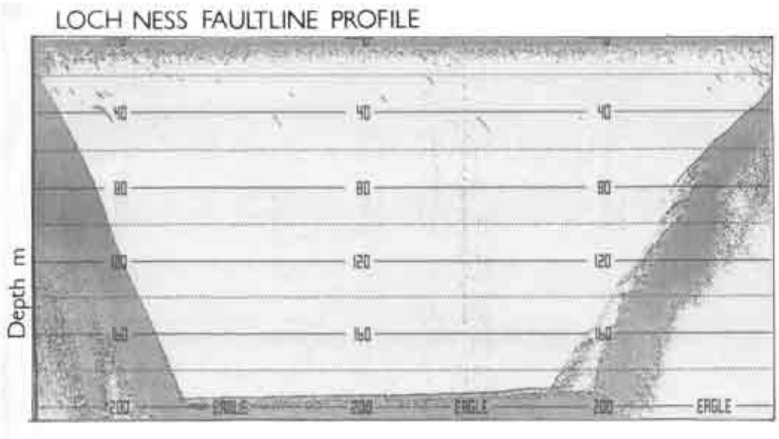


Figure 1: Transverse profile of the bathymetry of Loch Ness. At the surface, the Loch is 2 km wide. Model the sloping side walls as being 0.5 km wide each.

the correction

$$-\left(\frac{g}{c_s}\right)^2 \approx -\left(\frac{10}{1500}\right)^2 \approx -44 \times 10^{-6}\text{rad}^2\ s^{-2}.$$

This yields a substantial correction, and shows a better estimate of $N \approx 2.5 \times 10^{-3}$rad $s^{-1}$.

3 (4pt) In a Cartesian $x, y, z$ frame of reference with velocity vector $u, v, w$, perturbation pressure $p = (p_* - p_0)/\rho_*$ and perturbation buoyancy $b = -g\rho'/\rho_*$, related to perturbation density $\rho'$ scaled with uniform reference density $\rho_*$, the linearized, inviscid equations governing perturbations (internal waves of frequency $\omega$) of this uniformly-stratified fluid are, in Boussinesq-approximation, given by:

$$u_t = -p_x$$

$$v_t = -p_y$$

$$w_t = -p_z + b$$

$$b_t + wN^2 = 0$$

$$u_x + v_y + w_z = 0.$$

Here subscript-derivative notation is used, $z$ points antiparallel to gravity and $x$ and $y$ point in the along and across Loch directions respectively.

Derive the equation that governs the spatial structure for free internal waves propagating strictly in the transverse $(y, z)$ plane.

Using $\partial_x = 0$ and thus $u = 0$, waves are determined by adding cross-derivatives of the time-derivatives of the two remaining momentum equations and then using the buoyancy equation to eliminate buoyancy derivative $b_t$. This leads to

$$v_{ztt} - w_{ytt} - N^2 w_y = 0.$$

Introducing a streamfunction $v = \psi_z$, $w = -\psi_y$, this equation simplifies to

$$\frac{\partial^2}{\partial t^2}(\psi_{zz} + \psi_{yy}) + N^2 \psi_{yy} = 0.$$

4 (1pt) Consider monochromatic, plane waves and show that the buoyancy frequency acts as a high-frequency cut-off.

Consider $\psi \propto e^{i(ly+mz-\omega t)}$, with wave vector $\mathbf{k} = (l, m) = \kappa(\cos\alpha, \sin\alpha)$. Then, these waves need to satisfy

$$\omega^2 = N^2 \frac{l^2}{l^2 + m^2} = N^2 \cos^2 \alpha.$$

For real wave numbers $(l, m)$, the frequency is thus limited by $\omega \leq N$.

5 (1pt) What happens to perturbations of frequency higher than this cut-off?

For waves of frequency $\omega > N$, one of the wave numbers needs to be imaginary, and hence, is trapped to a boundary. Given that only the surface is free and can support waves, this boundary needs to be the surface, hence $m = iM$ (for $M$ real).

6 (2pt) What boundary conditions do these waves need to satisfy on the rigid sloping walls, bottom and free surface?

The solid walls need to be impenetrable $\mathbf{u} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ denotes a unit vector normal. to the boundary. This translates to a requirement that $\psi = \textit{constant}$ at these walls. At the free surface, $z = \zeta(y, t)$, in linear description the kinematic boundary condition reads $w = \zeta_t$. This has to be combined with the dynamic boundary condition which (for this scaled pressure) reads $p = g\zeta$. Using the horizontal momentum equation to eliminate the pressure, leads to $\psi_{ztt} = -g\psi_{yy}$. Using scales

$$[y, z, t] = [L, H, \omega^{-1}],$$

where $y = [y]y'$ etc, this equation can for monochromatic waves of frequency $\omega$, dropping primes, be simplified to

$$-\epsilon\psi_z + \psi_{yy} = 0, \quad \epsilon \equiv \frac{(L\omega)^2}{gH} < 0.1,$$

taking scales $L = 2km$, $\omega < N = 5 \times 10^{-3}$ rad s$^{-1}$, $H = 200$ m. Hence we approximate the surface boundary condition by $\psi_y = w = constant$. For reasons of continuity at the two sides, that constant has to be taken equal to zero, so that $\psi = 0$ also at the surface (rigid-lid approximation).

7 (1pt) Discuss the scaling by means of which the spatial structure of monochromatic internal waves propagating in cross-Loch direction is governed by the dimensionless equation

$$\psi_{yy} - \psi_{zz} = 0,$$

in terms of the stream function field $\psi(y, z)$.

Scaling the horizontal coordinate $y = Ly'$ by half-width $L$ $(= 1\ km)$, and the vertical coordinate $z = Dz'$ by scale height

$$D = L\left(\frac{N^2}{\omega^2} - 1\right)^{-1/2},$$

dropping primes, this leads to the required equation.

8 (1pt) In terms of these scaled variables, determine the location of the boundaries and boundary conditions.

For $y \in (-1/2, 1/2)$, the scaled, dimensionless bottom lies at

$$z = -\tau = -\frac{H}{D} = -\frac{H}{L}\left(\frac{N^2}{\omega^2} - 1\right)^{1/2},$$

the surface at $z = 0$, and the sloping walls at $z = 2\tau(y-1)$ for $y \in (1/2, 1)$ and $z = -2\tau(1+y)$ for $y \in (-1, -1/2)$.

9 (1pt) Discuss why internal gravity waves that propagate in transverse $y$-direction would be focused onto wave attractors.

The sloping walls lead to focusing reflections because wave beams, that follow the inclination set by the dispersion relation, is fixed, hence these beams decrease their width when falling onto a sloping wall from above. They increase their width due to defocusing, when incident onto such a slope from below. The reason why focusing dominates over defocusing lies in the fact that the cross-section of the former waves incident from above, is larger than that of the latter.

10 (2pt) Compute the range of frequencies over which 'simple' wave attractors exist, that is, having two surface reflections and one reflection from each side wall (use graphs to support your argument).(You get 0.5pt if you compute one frequency that falls within this range)

The range of (2,1) attractors, having 2 surface reflections and 1 reflection from each side wall, is on the one hand delimited by a characteristic web connecting a surface corner to the other surface corner, and on the other hand by the two bottom corners connecting to each other. This is equivalent to a characteristic from one corner reflecting either at the surface or bottom center. In the former case, characteristic $z = y - 1$ through $(1, 0)$ must thus cross $(0, -\tau)$, in the latter case, characteristic $z = -\tau - (y - 1/2)$ through $(1/2, -\tau)$ must cross $(0, 0)$. This leads to $\tau = 1$ and $\tau = 1/2$ respectively, with a corresponding frequency range from $\tau(\omega)$.

11 (1pt) Are there any transversely propagating internal waves that do *not* focus onto an attractor?

Yes, for waves having an inclination such that a characteristic starting in a surface corner point connects to a diagonally opposite bottom corner point. In this case, beams have the same amount of defocusing reflections as focusing reflections from the sloping sides. In that case the characteristic $z = y - 1$, passing through corner $(1, 0)$, has to cross the diagonally opposite corner $(-1/2, -\tau)$, which yields $\tau = 3/2$. The frequency that corresponds to this is given by $\omega = N/\sqrt{1 + (3L/2H)^2}$, where here $L$ represents half-width, $L = 1\ km$, so that $\omega \approx 0.13 \times N$.

12 (1 pt) For internal waves that also propagate down-channel, give a qualitative argument why internal gravity waves might still be trapped.

Internal wave beams then reflect from the sloping sides under an inclination with the $y$-direction, this leads to focusing and amplification of the cross-channel part of the wave (both of its velocity as well as the wave number), while the along-channel part does not change. This creates refraction of the wave, which leads more and more to cross-channel propagation.

13 (2pt) In *fall*, due to wind mixing and sheared bottom currents the water is well-mixed near surface and bottom, in the upper and lowest 75 m of the Loch. The temperature jump between the cold bottom and warm surface layer is thus restricted to the middle 50 m. Discuss qualitatively in what way this affects the internal gravity waves within the whole water column?

The internal waves can now propagate only within the stratified layer. It has evanescent tails in the homogeneous top and bottom layers. The waves are still subject to focusing, but now propagate with an effective reduced depth $H_{eff} = 50m$ instead of the actual 200 m depth.

14 (2pt) In *winter*, the entire water column is well-mixed, so that the Loch is homogeneous

in density. Now consider that the Loch is on a rotating planet, rotating at angular velocity $\Omega = 7.2 \times 10^{-5} rad/s$. Why does the Loch still support *internal* waves, and what kind of waves?

It supports inertial waves, restored by Coriolis forces. In fact, the slope also allows for nondivergent Rossby waves.

15 (4pt) In traditional approximation, the equations governing these perturbations (waves) are now given by

$$
\begin{aligned}
u_t - fv &= -p_x \\
v_t + fu &= -p_y \\
w_t - &= -p_z \\
u_x + v_y + w_z &= 0,
\end{aligned}
$$

where Coriolis frequency $f = 2\Omega \sin\phi$.

Taking into account the small aspect ratio of the Loch (depth 200 m, width 2 km and length 37 km), use scaling to write down approximate equations governing linear *surface* wave motions in this homogeneous-density fluid?

Use of the scales

$$[x, y, z, t, u, v, w, p] = [L, L, H, f^{-1}, U, U, UH/L, UfL],$$

with $x = [x]x'$ etc and dropping primes afterwards, leads to:

$$
\begin{aligned}
u_t - v &= -p_x \\
v_t + u &= -p_y \\
\delta^2 w_t - &= -p_z \\
u_x + v_y + w_z &= 0,
\end{aligned}
$$

where $\delta = H/L = 0.1 \ll 1$. Expanding each dependent variable in a perturbation series, e.g. $u = u^{(0)} + \delta u^{(1)} + \cdots$, and collecting terms to order zero in $\delta$ shows that the pressure and hence

$u, v$ must be $z$-independent. Hence, the continuity equation shows $w = (z/H + 1)\zeta_t$, obeying the kinematic boundary condition at the free surface and vanishing correctly at the bottom, $z = -H$. Pressure scale $UfL$ now equals the pressure scale due to free surface displacement, $g\zeta$ which reads $g[w]/f = gHU/fL$. This requires length scale, L, to be identical to the Rossby deformation radius $R = \sqrt{gH}/f$. Nondimensionally this leads to $p = \zeta$. Thus we obtain the rotating shallow-water equations

$$u_t - v = -\zeta_x$$
$$v_t + u = -\zeta_y$$
$$\zeta_t + u_x + v_y = 0,$$

16 (1pt) On what grounds can rotational effects on *surface* gravity waves be neglected?

This is based on the fact that rotational effects on surface gravity waves are relevant for waves of scales larger than the Rossby deformation radius, $R \equiv \sqrt{gH}/f \approx \sqrt{10 \times 200} \times 10^4\ m \approx 45$ km, which is much larger than the width as well the length of the Loch.

17 (1pt) Now further idealize the cross-sectional shape of the Loch to a rectangle (of width 1.5 km) and assume the Loch to be infinitely long. Compute *internal* wave solutions for this homogeneous-density rotating fluid, assuming these waves propagate in along-Loch $x$-direction.

By cross-differentiation one shows all variables satisfy the same equation

$$\mathcal{L}(u, v, w, p) = 0, \text{ where } \mathcal{L} = (\partial_{xx} + \partial_{yy} + \partial_{zz})\partial_{tt} + \partial_{zz}$$

Applying this to the transverse velocity, $v$, that is required to vanish at the sides, shows $v \propto \sin(m\pi y)$. The vertical dependence must be $\propto \sin(m\pi y)$ and the along-Loch dependence $\propto e^{i(kx-\omega t)}$. Hence solutions have a transverse velocity field

$$v = V \sin(m\pi y) \sin(n\pi z) \cos(kx - \omega t),$$

while $\omega$ must satisfy the dispersion relation detailed in (18).

18 (2pt) What are the Loch's transverse eigenfrequencies, if any? Can there be any attractors in this case? Why, or why not?

Inserting this into the governing equation shows

$$\omega^2 = \frac{n^2\pi^2}{k^2 + m^2\pi^2 + n^2\pi^2}.$$

Yes, provided the non-traditional Coriolis term is also taken into account.

19 (2pt) What is the difference between *inertial oscillations* and *inertial waves*?

Inertial oscillations are non-propagating anticyclonic horizontal circular motions, having infinite spatial scales and frequency $\omega = f$. Inertial *waves* have finite wave length and are propagating both horizontally as well as vertically, and exist over the frequency range $\omega \in (0, f)$. They also perform circular motions, but in an inclined plane.

20 (2pt) Now assume stratification and rotation are both present. In a uniformly-stratified ($N = const$), rotating fluid, in which rotation axis $\boldsymbol{\Omega}$ is anti-parallel to the direction of gravity $\mathbf{g}$, plane monochromatic internal waves of frequency $\omega$ satisfy a dispersion relation given by

$$\omega^2 = N^2\cos^2\alpha + f^2\sin^2\alpha = \frac{N^2k^2 + f^2m^2}{k^2 + m^2},$$

where 2D wave vector $\mathbf{k} = (k, m) = \kappa(\cos\alpha, \sin\alpha)$.

Compute the phase and group velocity vectors of these waves and show that they are orthogonal to one another?

With

$$\omega^2 = N^2\cos^2\alpha + f^2\sin^2\alpha = N^2\frac{l^2}{l^2 + m^2} + f^2\frac{m^2}{l^2 + m^2}$$

Phase velocity vector $\mathbf{c} = \frac{\omega}{\kappa^2}\mathbf{k}$, and group velocity vector

$$\mathbf{c_g} = \nabla_{\mathbf{k}}\omega = \frac{1}{2\omega}\nabla_{\mathbf{k}}\omega^2 = \frac{(N^2 - F^2)km}{\omega(k^2 + m^2)^2}(m, -k).$$

This is obviously orthogonal to the phase velocity vector as $\mathbf{c_g} \cdot \mathbf{c} = 0$.