# Exam Fluid Dynamics

April 14, 2023

Total number of points: 100 (+10 Bonus)
You can answer in English or Dutch, but please be consistent.
Subquestions are largely independent of each other.
Notation: $\partial_x f = \partial f / \partial x$ for any function $f$ and variable $x$.

The first question (40 points) is an application of frictionless flow (shallow-water waves).
The second question (20 points) is about viscous flow (pipe flow).
Questions 1 and 2 follow quite closely material from the lectures and tutorials.
The third equation (40 points) is a bit more qualitative and is about thermodynamic effects and stability.
Some parts of question 3 require a little out-of-the-box thinking.

## Question 1 - Tsunami as a shallow-water wave (40)

Tsunamis are gravity waves which arise after submarine earthquakes or landslides. In this question, we estimate the time a tsunami takes to cross the ocean.

- We assume that the tsunami can be described as linear shallow-water wave (not a great approximation). "Shallow water" implies that the water depth is much smaller than the length scale at which horizontal velocities vary, therefore horizontal velocity is approximately constant over the whole water column, $\partial_z u = 0$.
- We assume that the wave can be described as 2D (one vertical, one horizontal direction).
- The unperturbed ocean surface is at $z = 0$, the ocean floor is at constant depth $z = -H$. The actual (perturbed) sea surface is at $z = \eta(x,t)$.
- Sea water is assumed to have constant density $\rho = 1000 kg/m^3$. Frictional forces are ignored (zero viscosity, $\mu = \lambda = 0$). Surface tension is also ignored. The only volume force is gravity, with gravitational acceleration $g \approx 10 m/s$.
- Air pressure is constant $p_0$ at $z = \eta$.
- Observations suggest that a reasonable order of magnitude for the wave period is about $T = 1000s$ and for the wavelength $\Lambda = 100km$.
- The geometry is given in fig. 1.

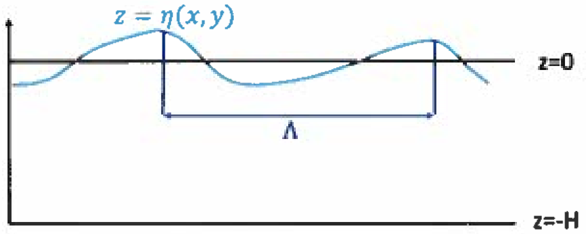



Figure 1: The geometry for question 1.

a (7) Explain briefly why the governing equations incl. boundary conditions in the water become:

$$\begin{aligned}
\rho(\partial_t u + u\partial_x u + w\partial_z u) &= -\partial_x p \\
\rho(\partial_t w + u\partial_x w + w\partial_z w) &= -\partial_z p - \rho g \\
\partial_x u + \partial_z w &= 0 \\
w &= \partial_t \eta + u\partial_x \eta = D\eta/dt \quad (z = \eta) \\
p &= p_0 \quad (z = \eta) \\
w &= 0 \quad (z = -H)
\end{aligned}$$

Briefly comment on why we do not assume a no-slip condition at the bottom ($z = -H$).

b (6) In the following, we assume that the amplitude of the tsunami is "small". We do not know a priori what this means, but make a preliminary guess that an acceptable upper estimate for the order of magnitude of the water velocity (both vertical and horizontal) is at most $U = 1m/s$.

Show that, to a good approximation, we have hydrostatic balance,

$$p(x, z, t) = p_0 + \rho g \eta(x, t) - \rho g z$$

c (6) Next, show that

$$D\eta/dt = -(H + \eta)\partial_x u$$

Give a (brief!) physical interpretation of this equation.
Hint: Use the continuity equation, recall the shallow water assumption mentioned above, integrate vertically from $z = 0$ to $z = \eta$.

d (5) Show that, for $u, \eta$ "small enough", the momentum equations become

$$\begin{aligned}
\partial_t \eta &= H\partial_x u \\
\partial_t u &= -g\partial_x \eta
\end{aligned}$$

e (5) Show that

$$\partial_t^2 \eta = gH\partial_x^2 \eta$$

f (5) We make an inspired guess that $\eta = \eta_0 \sin(kx - \Omega t)$ is a solution. Show (by simple checking) that this guess indeed is a solution provided that $\Omega = \pm\sqrt{(gH)}k$.

g (6) Use f) to show that the the phase speed (the speed at which wave crests travel) is $c = \sqrt{gH}$. Use the result to estimate the time a wave needs to cross an ocean of 4km depth and 10000km wide.

# Question 2 - Viscous pipe flow (20)

A viscous fluid of (constant) density $\rho$ flows through a long, horizontal pipe of radius $R$ under the influence of a constant horizontal pressure gradient $\beta$. No volume forces exist in the horizontal direction. The geometry is given in fig. 2.

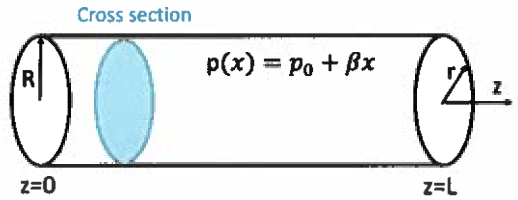



Figure 2: The geometry for question 2, including a cross section through the pipe (perpendicular to the axial direction).

a (5) Using a suitable coordinate system and making suitable assumptions regarding symmetry (briefly name these), show that the relevant momentum balance (far enough from the end points of the pipe) can be written as:

$$0 = -\beta + \mu \frac{d^2 w}{dr^2} + \mu/r \; \frac{dw}{dr} \qquad (1)$$

where the velocity in z-direction, $w$, only depends on $r$.

b (7) State a suitable boundary condition for $w$ and derive the velocity profile,

$$w(r) = \beta/4\mu \; (r^2 - R^2)$$

Hint: Try a test function $dw/dr \equiv g(r) = \gamma r^n$.

If you cannot actually *derive* this from equation (1), you will get 4/7 points by *checking* (through simply inserting and working out) that the solution fulfills equation (1) and the boundary condition.

c (3) Argue that mass flux $Q$ through a cross section of the pipe is proportional to $-\beta$, i.e. minus the pressure gradient. You can either compute $Q$ or give a simple physical argument.

d (5) The fluid is water ($\mu = 10^{-3} kg/ms$, $\rho = 1000 m/s^3$), the pressure gradient is $0.4 N/m^2$, and the pipe radius is $R = 1cm$. Are your assumptions regarding symmetry in question a) justified? Hint: In his famous experiment, Reynolds observed turbulence for $Re > 2000$.

$$\Phi = \int_0^R \int_0^{2\pi} \frac{\beta}{4\mu}(r^2 - R^2)\, d\theta\, dr$$

$$= 2\pi \frac{\beta}{4\mu}\left(\frac{1}{3}r^3 - R^2\right)\Big|_0^R = \frac{\pi\beta}{2\mu}\left(\frac{1}{3}R^3 - R^2 + R^2\right)$$

$$=$$

## Question 3 - Temperature-driven and salt-driven Atlantic circulation (40)

In the North Atlantic, warm salty water flows near the surface from the equator to the Northpole ("the Gulfstream"), and cold fresher water flows back from the Northpole to the equator in deeper layers. The Stommel model is a very simple model of this dynamics and can in principle be built in a lab.

- The model consists of two boxes filled with water, the "polar" and "equatorial" box. Each box contains a mixer which ensures that the temperature $\vartheta$ and salinity (salt concentration) $S$ are homogenous inside the box.
- Density is given by $\rho = \rho_0(1 - \alpha_\vartheta(\vartheta - \vartheta_0) + \alpha_s(S - S_0))$ with $\alpha_\vartheta$ and $\alpha_S$ positive.
- The boxes are connected by two horizontal cylindrical pipes of length $L$ at height $z = -H_u$ (upper pipe) and $z = -H_l$ (lower pipe), with $H_u \ll H_l$. As shown in question 2, the flow through each pipe is proportional to the pressure gradient $\beta$ in the pipe; we assume $Q_l = K\beta_l$ and $Q_u = K\beta_u$.
- The walls of the pipes and the side walls and bottom of the boxes are made material with perfect thermal isolation ($k_w = 0$). However, each box has a lid which moves freely up and down with the water level $\eta$ and has a finite thermal conductivity $k_l$, density $\rho_l$, heat capacity $C_{vl}$, and thickness $D$.
- The air temperature above the lid of the polar and the equatorial box is kept constant at $\vartheta_{0p}$ and $\vartheta_{0e}$, respectively, with $\vartheta_{0e} > \vartheta_{0p}$.
- Finally, there is some device to regulate the salinity in each box. In questions a-f we assume that salinity is constant and equal in both boxes ($S_e = S_p = S_0$).
- The geometry is given in fig. 3.

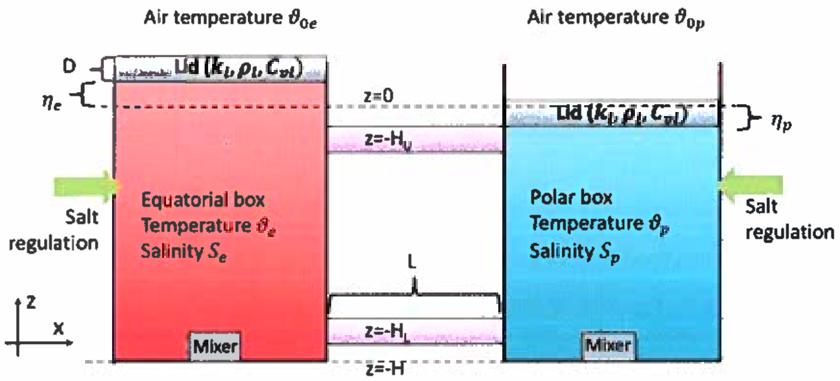



Figure 3: The geometry for question 3 (Stommel model).

a (5) As shown in question 2, the mass flux through each pipe is proportional to the pressure gradient $\beta$ in the pipe: we assume $Q_l = -K\beta_l$ and $Q_u = -K\beta_u$. Here $K$ is a positive constant, and assumed to be independent of, e.g., the temperature in the pipe.

Show that $Q_l = -Kg/L\ (\rho_p\eta_p - \rho_e\eta_e + H_l(\rho_p - \rho_e))$.
In which direction does the current flow if

- $\vartheta_e = \vartheta_0$ but $\eta_e > \eta_p$?
- $\vartheta_e > \vartheta_0$ but $\eta_e = \eta_p$?

Hint: You may use hydrostatic balance, $p(z) = p_0 + \rho g(\eta - z)$ (no need to derive this).

b (7) Assume that $\vartheta_p$ and $\vartheta_{0p}$ are fixed, and no heat sources exist in the lid. Horizontal dependence (due to effects of the side edges of the lids) can be ignored. Show that the (stationary) temperature profile in the lid is $\vartheta_l(z) = \vartheta_p + (\vartheta_{0p} - \vartheta_p)(z - \eta_p)/D$.

Hints: Intermediate result: $d^2\vartheta/dz^2 = 0$. Use $\tilde{z} = z - \eta_p$ and define boundary conditions at suitable values of $\tilde{z}$.

c (3) Under the same assumptions as in b), derive the heat flux through the lid.
(Note: you will find that the larger the temperature difference between air and water, the stronger the heat flux ... and hence the heat loss or gain in the water.)

d (7) Assume that initially, $\eta_p = \eta_e = 0$ and $\vartheta_p = \vartheta_{0p}$ and $\vartheta_e = \vartheta_{0e}$. Also, initially $Q_l = Q_u = 0$.
Eventually, a convection-like circulation will arise, with $Q_u$ and $Q_l$ non-zero and of opposite sign. Qualitatively describe the physical processes which cause the circulation and explain whether $Q_l$ or $Q_u$ is positive (to the right).

Using results from b) and c) it can be shown (don't do this!) that the Stommel model, still with $S_e = S_p = S_0$, behaves as follows:

$$\begin{aligned} d\Theta/dt &= -|Q(t)|\Theta(t) - \gamma_1(\Theta(t) - \Theta_0) \\ Q(t) &= \gamma_2\Theta(t) \end{aligned}$$

where $\Theta(t) \propto (\vartheta_e - \vartheta_p)$ is the temperature difference between the equatorial and polar box (up to a scaling factor), and $\Theta_0 \propto (\vartheta_{0e} - \vartheta_{0p}) > 0$ is the difference between the air temperature above the two boxes (again, up to a scaling factor).
$Q = Q_u = -Q_l$ is the strength of the circulation.
Finally, $\gamma_1, \gamma_2$ are positive constants.
The term $-|Q(t)|\Theta(t)$ describes the effect of mixing warmer and colder fluid because of flow through the pipes.
The term $-\gamma_1(\Theta(t) - \Theta_0)$ is the effect of unequal air temperature above the two lids, which tries to maintain the same temperature difference in the water.

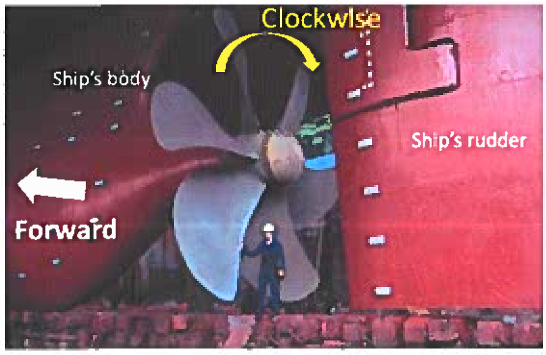
e (5) Show that $\bar{\Theta} = -\gamma_1/2\gamma_2 + \sqrt{\gamma_1^2/4\gamma_2^2 + \gamma_1\Theta_0/\gamma_2}$ is an equilibrium solution of the Stommel model.
Hint: Formulate a condition for equilibrium solutions to the Stommel model. Check that with the suggested expression for $\bar{\Theta}$, $Q$ is always positive (for $\Theta_0 > 0$). Then show that the suggested expression $\bar{\Theta}$ is indeed a solution. For the last step, inserting and checking is sufficient.

f (5) Finally, show that the equilibrium solution $\bar{\Theta}$ is stable against small perturbations.
Hint: Intermediate result: $d\theta/dt = -2\gamma_2\bar{\Theta}\theta - \gamma_1\theta$.

g (5) Suppose that we now keep the temperature fixed ($\vartheta_p = \vartheta_e = \vartheta_0$) but regulate the salinity such that $S_p < S_e$. How will the circulation look like, in particular the sign of $Q_l$ and $Q_u$? Give a brief qualitative argument.
Hint: density!

h (3) Suppose that the Stommel model represents the North Atlantic circulation well. What might happen to the Gulfstream if ice melt in Greenland causes a massive inflow of fresh (i.e. low-salinity) water in the polar regions?

## Bonus Question - Ship screw (10)



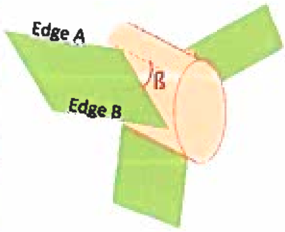



Figure 4: The geometry of a propellor.

In fig. 4, a ship's propellor is shown. Suppose that the propellor rotates clockwise (yellow curly arrow), does the ship then move forward (white straight arrow) or backward? Give a brief argument. You may use formulae, but you do not need to do any actual calculations.

Hint: For simplicity, you can think of the propellor blades as flat plates rectangular which are tilted by an angle $\beta$ with respect to the axial direction.

Up to 5 bonus points will be given for sketching a good approach how to solve this, even if a final answer is lacking.