Fluid dynamics exam 12.4. 2024

April 12, 2024

Instructions:

- Solve each question on a new sheet (bonus may go on the same sheet as question 1). Write your name on every sheet!
- Calculations: it is fine to round ($g \approx 10m/s^2$, $\pi \approx 3$) as long as it is clear what you do.
- Make sure to write correct units: E.g. the gravitational force on 1kg of mass is $F = Mg = 1kg * 10m/s^2 = 10N$. Writing $F = Mg = 1 * 10 = 10N$ is wrong.
- Quite some useful stuff is available in the appendix!
- If you get stuck with a subquestion, you can still solve the next subquestion.
- Statements like "max 3 sentences" serve to help you judge the amount of precision required to get full points. Don't spend your time lengthy answers or mathematical derivations where it's not needed. A small sketch is always allowed, if helpful to make your point.
- There are 100 points + 10 bonus points.

All the best!!

Question 1 - the Kentucky Fried Chicken Airplane (30 points)

Youtuber PeterSripol built a model airplane with wings from Kentucky Fried Chicken buckets (see fig. 1)[1]. The wings can rotate around the horizontal axis.

We assume that we can treat the two wings as one cylinder and that the flow around the cylinder is 2D (we ignore the effect of the ends of the cylinder on the flow). Also, we introduce a coordinate system where x=0,y=0 along the symmetry axis of the wings.

a (3) Name three assumptions under which we can model the airplane by using the following form of Bernoulli's theorem:

$$v^2/2 + p/\rho = const \text{ (in space and time)}.$$

No need for derivations - naming the conditions is enough.

In the following we assume that Bernoulli can indeed be used, at least approximately.

b (5) A (real) streamfunction is defined by $(u, v) = (\partial_y \psi, -\partial_x \psi)$.

Argue that the streamfunction must be constant along the surface of the wing, in other words, for $x = R\cos(\gamma), y = R\sin(\gamma)$ (where R is the wing's radius) we get $\psi(R\cos(\gamma), R\sin(\gamma)) = const$.

Hint: Think of boundary conditions for the velocity on the surface.

c (5) In the lecture we have seen that we can describe the flow around a circle (or cylinder) by using the sum of a parallel flow and a dipole flow. Assume the airplane to fly towards the negative x-direction, i.e. from the airplane's perspective, the wind far from the wings is towards the positive x-direction, with velocity U.

[1] source: https://www.youtube.com/watch?v=K6geOms33Dk

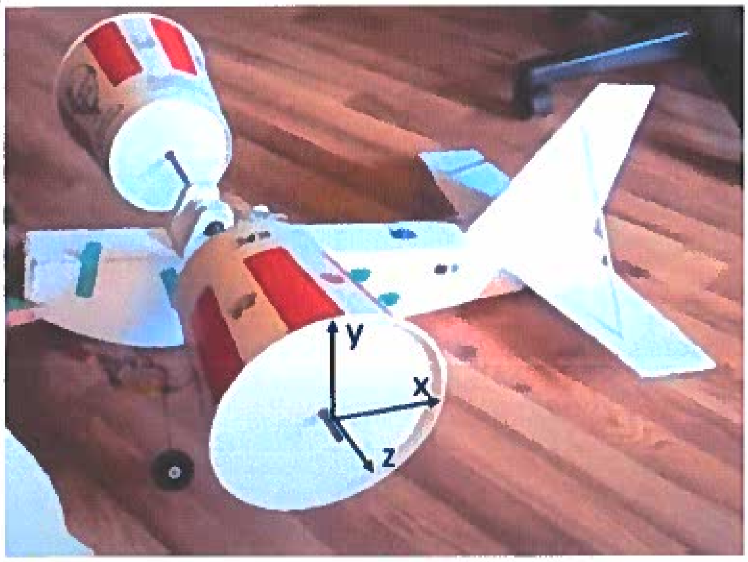

Figure 1: Picture of the Kentucky Fried Chicken airplane with a coordinate system centred at the wing axis.

For a radius R of the wings and velocity U, which value for the dipole strength μ must be chosen such as to fulfill the boundary condition outlined in b)? Hint: use the appendix...

d (6) In which direction should the wings rotate in order to generate upward lift? Give a qualitative physical argument (max 6 sentences + 1 sketch).

e (5) The radius of the "wings" is $R = 10cm$, and they rotate with about 15 rotations per second, so the angular frequency is $\omega = 100s^{-1}$.

Show that one would expect for the absolute value of the circulation: $|\Gamma| \approx 6m^2/s$ (rounding is allowed!).

Are there effects that could prove us wrong? (max 3 sentences)

f (6) The Kentucky Fried Chicken Airplane has the following geometry: The radius of the "wings" is $R = 10cm$, the length of the two wings combined is $L = 50cm$, and the whole airplane weighs 300g. The density of air is about $1kg/m^3$ and the gravitational acceleration is about $10N/kg$.

Assuming that the rotation of the wings indeed introduces a circulation with a absolute value of $|\Gamma| = 6\frac{m^2}{s}$, what is the minimal velocity needed to keep the airplane flying?

Question 2 - lava flow (35 points)

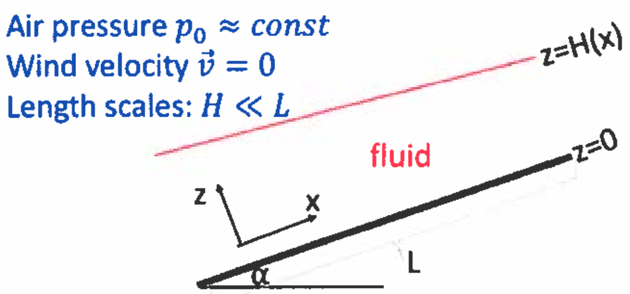

Figure 2: Viscous fluid flowing down a hill with constant slope under constant air pressure and no wind.

A thin sheet of a very viscous fluid with constant density ρ flows down a hill with a constant slope α, see fig. 2.

We choose a coordinate system such that z is the normal vector to the slope, x is a tangential vector pointing uphill, and y a tangential vector perpendicular to x. The sheet is very broad and uniform in the y-direction. The thinkness of the sheet is H and may depend on x(and t). The length L of the hill is very long (so $H \ll L$), and we can ignore the edges of the hill. The air pressure at $z = H$ is p_0 and is approximately constant in t, x, y (our hill is not very high). Air is several orders of magnitude less viscous than the fluid, so we put $\mu_{air} \approx 0$.

We assume that the governing equations become:

$$\begin{aligned}
\partial_x u + \partial_z w &= 0 \\
\rho(\partial_t u + u\partial_x u + w\partial_z u) &= -\partial_x p - g\rho\sin(\alpha) + \mu(\partial_x^2 u + \partial_z^2 u) \\
\rho(\partial_t w + u\partial_x w + w\partial_z w) &= -\partial_z p - g\rho\cos(\alpha) + \mu(\partial_x^2 w + \partial_z^2 w)
\end{aligned} \tag{1}$$

where u and w are the velocities in the x and z directions, respectively.

a (4) Briefly explain three simplifying assumptions that have been made to simplify the original continuity and momentum balance equations to achieve this result (max. 3 sentences per assumption). In addition, briefly explain why the gravity term takes the form used here (max 3 sentences).

b (2) Explain briefly (max 3 sentences) why the following boundary conditions can be assumed at the bottom ($z = 0$):

$$\begin{aligned}
w(x, z = 0) &= 0 \\
u(x, z = 0) &= 0
\end{aligned}$$

c (4) Use one of the basic equations in (1) and scale analysis to show that w is small. In addition, show that if the situation is translation-invariant in x (in particular, x- derivatives of the velocity vanish), then $w = 0$.

d (4) We will use the following boundary conditions at the top ($z = H(t, x)$):

$$\begin{aligned}
p(x, H) &= p_0 \\
w(H) &= \partial_t H + u\partial_x H \\
\partial_z u|_{z=H} &= 0
\end{aligned}$$

For each of these equations, name the boundary condition from which it has been derived (no need to actually do the derivation!).

Briefly explain 3 assumptions (based on earlier results and/or additional ones) that were made to arrive at these simplified forms. Max 2 sentences per assumption, no need to do mathematical derivations.

From now on, we make the additional assumption that the situation is stationary, hence ∂_t-terms drop.

e (5) Use scale analysis to argue that all advection terms $v_i\partial_i v_j$ can be ignored if $ReH/L \ll 1$ is small, where the Reynolds number is given by $Re = UH\rho/\mu$. In addition, use scale analysis to argue that $\mu(\partial_x^2 v_i + \partial_z^2 v_i) \approx \mu(\partial_z^2 v_i)$ for $v_i \in \{u, w\}$.

With the results from e) the governing equations thus become:

$$\begin{aligned}
\partial_x u + \partial_z w &= 0 \\
0 &= -\partial_x p - g\rho\sin(\alpha) + \mu(\partial_z^2 u) \\
0 &= -\partial_z p - g\rho\cos(\alpha) + \mu(\partial_z^2 w)
\end{aligned} \tag{2}$$

with the boundary conditions

$$w(x, z = 0) = 0$$

$$
\begin{aligned}
u(x, z=0) &= 0 \\
p(x, H) &= p_0 \\
w(H) &= u\partial_x H \\
\partial_z u|_{z=H} &= 0
\end{aligned}
$$

f (5) We now assume translation invariance in x, hence $w = 0$ (see c) and $H = const$.
Solve for p.
Show that $u(z) = Az^2/2 - AHz$ where $A = (\partial_x p + g\rho\sin(\alpha))/\mu$. Note: under our assumptions, $\partial_x p$ in fact equals zero.

g (5) The viscous material is basaltic lava. Measurements show that $H = 20cm$, and the velocity at the surface of the lava stream is $0.6m/s$. The density of the lava is $\rho = 3000kg/m^3$ and its temperature is 1200^oC. The slope of the hill is $\alpha = 6°$.
What is the viscosity of the lava? (Some useful values are in the appendix.)

h (6) We now relax the assumption of translational invariance in x, and assume that p, H, μ, u, w can vary slowly in x.
The volume flux (per unit length in y) of the lava is given by $Q = \int_0^H u\, dz$.
Argue why $\partial_x Q = 0$ in a stationary situation.

When lava cools, it becomes more viscous. In our case, the further the lava has flowed towards the negative x-direction, the more viscous it is. Will H increase or decrease towards the downhill (negative x) direction? Give a brief qualitative argument.

Question 3 - Isolating your house (35 points)

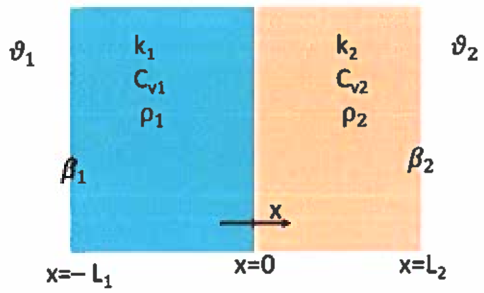

Figure 3: Sketch of a wall with two layers.

In this question we look at a wall that is (nearly) infinitely large and homogenous in the y and z direction. The wall consists of two materials, each of which has its own (constant) density ρ_i, heat conductivity k_i and heat capacity at constant volume C_{v_i}, see fig. 3. Here $i \in \{1, 2\}$. At the inside of the wall (inside the house), $x = +L_2$, the air temperature is held at a constant value ϑ_2, but outside, at $x = -L_1$, the temperature is ϑ_1 , which can vary in time depending on the time of the day. Also, the heat transfer coefficient between wall and air at $x = +L_2$ is β_2 , i.e. the heat flux towards the positive x-direction between wall and air is

$$\mathbf{q}_2 = -\beta_2(\vartheta_2 - \vartheta(L_2))\mathbf{e}_\mathbf{x}$$

where $\vartheta(L_2)$ is the temperature just inside the wall and $\mathbf{e}_\mathbf{x}$ the unit vector in x-direction. In other words, the heat flux from wall to air is driven by the temperature difference between wall and air. Similarly, the heat flux between wall and air at $x = -L_1$ is given by

$$\mathbf{q}_1 = -\beta_1(\vartheta(-L_1) - \vartheta_1)\mathbf{e}_\mathbf{x}$$

The governing equations for temperature ϑ in a solid, and in absence of internal energy sources, are given by (no need to derive this!):

$$\rho C_v \partial_t \vartheta = -\nabla \cdot \mathbf{q}$$
$$\mathbf{q} = -k\nabla \cdot \vartheta$$

where $\mathbf{q}$ is the heat flux.

a (3) Derive that within each solid, $\partial_t \vartheta = \kappa \partial_x^2 \vartheta$ with $\kappa = k/(\rho C_v)$.

b (6) Argue that the following boundary conditions hold (max. 8 sentences in total):

$$\begin{aligned}
\vartheta|_{x=-\delta} &= \vartheta|_{x=\delta} && \delta \to 0 \\
k_1 \partial_x \vartheta|_{x=-\delta} &= k_2 \partial_x \vartheta|_{x=\delta} && \delta \to 0 \\
k_2 \partial_x \vartheta|_{x=L_2} &= \beta_2(\vartheta_2 - \vartheta(L_2)) && \text{at } x = L_2 \\
k_1 \partial_x \vartheta|_{x=-L_1} &= \beta_1(\vartheta(-L_1) - \vartheta_1) && \text{at } x = -L_1
\end{aligned}$$

Notation: The second equation tells us that the temperature in the left block of material, just to the left of $x = 0$, equals the temperature in the right block, just to the right of $x = 0$.

c (3) Argue that for $\beta_1 \to \infty$, the boundary condition at $x = -L_1$ simplifies to $\vartheta(-L_1) = \vartheta_1$. Explain in max 2 sentences why this makes sense physically.

Note: Analogously, for $\beta_2 \to \infty$, we obtain $\vartheta(L_2) = \vartheta_2$. From now onward, we will assume that indeed both simplified conditions hold.

d (7) We now assume (in questions d-f) a stationary situation.
Using the results so far, show that

$$\begin{aligned}
\vartheta(x \le 0) &= ax + b && \text{for } x \le 0 \\
\vartheta(x \ge 0) &= cx + d && \text{for } x \ge 0
\end{aligned}$$

Also show that a and c are of the form given below. You do NOT have to show the results for b and d, but they are given for completeness.

$$\begin{aligned}
a &= (\vartheta_2 - \vartheta_1)/(k_1 L_2/k_2 + L_1) \\
b &= \vartheta_2 - (\vartheta_2 - \vartheta_1)/(1 + k_2 L_1/(k_1 L_2)) \\
c &= (\vartheta_2 - \vartheta_1)/(L_2 + k_2 L_1/k_1) \\
d &= \vartheta_2 - (\vartheta_2 - \vartheta_1)/(1 + k_2 L_1/(k_1 L_2))
\end{aligned}$$

e (3) Show that the heatflux through the wall is given by $q = -(\vartheta_2 - \vartheta_1)/(L_1/k_1 + L_2/k_2)$.

f (3) At first, consider a with walls consisting of a single layer, $L_1 = L$ and $L_2 = 0$ and k_1 be high (good heat conductor, bad isolator).

Now we isolate the walls by adding a layer such that $L_2 = L$ (same thickness as the first layer) and heat conductivity $k_2 = k_1/100$.

How much energy will roughly be saved by the isolation if the building is always heated to 20ºC while the outside temperature is 0ºC? (Ignore windows)

i) 99% of the value before isolation
ii) 90% of the value before isolation
iii) 10% of the value before isolation
iv) 1% of the value before isolation
Motivate your answer.

g (10) Now let's suppose that the outside temperature varies in time as $\vartheta_1(t)$ (which is given), while the inside temperature ϑ_2 is constant.

Figure 4: Sketch of a 1D discretisation (split into grid points) of the wall.

Suppose you have to determine the temperature $\vartheta(x,t)$ inside the wall numerically. Suppose that for a given time step t_k you have determined the temperature at all grid points x_j (grid points are a distance Δx apart, see fig. 4).

What is then the temperature one timestep later (at $t_{k+1} = t_k + \Delta t$) at a point x_j with $-L_1 < x_k < 0$, i.e. a point not in the boundary? Briefly make clear how you obtained this.

Also, describe how you would choose Δx and Δt (max 5 sentences).

Bonus question - Orr-Sommerfeld equations for plane Couette flow (10 points)

Consider a plane Couette flow, i.e. a flow between two horizontal plates of which one is motionless and the other moves at a fixed velocity, in the absence of a horizontal pressure gradient. In the lecture and the tutorial, we derived the Orr-Sommerfeld equations for very small perturbations from the Navier-Stokes equations by considering small perturbations to the base flow and linearising the equations.

The Orr-Sommerfeld equations for a plane Couette flow are stable for all Reynolds numbers. Nonetheless, experiments in the lab revealed that the flow is not stable, but can become turbulent. How can this be explained? (The explanation is NOT that the lab workers made a mistake with producing the plane Couette flow!)

xx Sofie ☺ Veel studeer succes!

Appendix

Basic equations in Cartesian coordinates

We use the notation $\mathbf{x} = (x, y, z)^T$ and $\mathbf{v} = (u, v, w)^T$ and

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\ \nabla \wedge \mathbf{v} &= (\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})^T \end{aligned}$$

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Momentum balance:

$$\begin{aligned} \rho\frac{Du}{dt} &= f_1 + \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial y} + \frac{\partial T_{13}}{\partial z} \\ \rho\frac{Dv}{dt} &= f_2 + \frac{\partial T_{21}}{\partial x} + \frac{\partial T_{22}}{\partial y} + \frac{\partial T_{23}}{\partial z} \\ \rho\frac{Dw}{dt} &= f_3 + \frac{\partial T_{31}}{\partial x} + \frac{\partial T_{32}}{\partial y} + \frac{\partial T_{33}}{\partial z} \end{aligned}$$

with the material derivative

$$\frac{D}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

and the stress tensor for a Newtonian fluid:

$$\begin{aligned} T_{11} &= -p + 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{v} \\ T_{22} &= -p + 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{v} \\ T_{33} &= -p + 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{v} \\ T_{12} = T_{21} &= \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ T_{13} = T_{31} &= \mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) \\ T_{23} = T_{32} &= \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \end{aligned}$$

The thermal energy balance is given by

$$\rho C_v\frac{D\vartheta}{dt} = -\nabla\cdot\mathbf{q} + \nabla\mathbf{v} : \mathbf{T} + \rho r$$

where the heat flux obeys

$$\mathbf{q} = -k\nabla\vartheta$$

and the work done by forces contributes as

$$\nabla\mathbf{v} : \mathbf{T} = -p\nabla\cdot\mathbf{v} + \mathcal{D}$$

where the dissipation function is given by

$$\mathcal{D} = 2\mu(\frac{\partial u}{\partial x}^2 + \frac{\partial v}{\partial y}^2 + \frac{\partial w}{\partial z}^2) + \mu((\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})^2 + (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})^2)$$

Boundary conditions

We define $\mathbf{n}$ to be the normal vector and $\mathbf{t}$ is a tangential vector.

At a wall: Kinematic boundary condition (solid boundary)

$$\mathbf{u} \cdot \mathbf{n} = 0$$

No-slip:

$$\mathbf{u} \cdot \mathbf{t} = 0$$

At a free surface: We assume that the surface is roughly perpendicular to the z-axis, but can deviate locally (like a water surface which is on average horizontal but has some waves). The surface is at $z = \eta(x, y, t)$.

Kinematic boundary condition:

$$v_z = \partial_t \eta + v_x \partial_x \eta + v_y \partial_y \eta \quad at\ z = \eta(x, y, t)$$

Dynamic boundary condition, normal direction:

$$p' - \lambda' \nabla \cdot \mathbf{v}' - \sum_{ij} \mu'(\partial_i v'_j + \partial_j v'_i) n_i n_j = p - \lambda \nabla \cdot \mathbf{v} - \sum_{ij} \mu(\partial_i v_j + \partial_j v_i) n_i n_j - \sigma(1/R_1 + 1/R_2)$$

where R_1, R_2 are positive if the fluid whose properties are denoted by plain letters (μ, p, v...) bulges into the fluid with primed letters μ', v', p').

Dynamic boundary condition, tangential direction:

$$\sum_{ij} (\mu(\partial_i v_j + \partial_j v_i) - \mu'(\partial_i v'_j + \partial_j v'_i)) n_j t_i = 0$$

Useful formulae for potential flows around a cylinder

Complex coordinates: $z = x + iy$

Complex potential: $f(z) = \zeta + i\psi$

Velocity potential ζ: $(u, v) = \nabla \zeta$

Streamfunction ψ: $(u, v) = (\partial_y \psi, -\partial_x \psi)$

Complex velocity: $u - iv = df/dz$

Complex potential of a parallel flow: $f(z) = Uz$ for $U = const$ (real if the velocity is parallel to the x-axis)

Complex potential of a dipole flow centered in the origin: $f(z) = -\frac{\mu}{2\pi}\frac{1}{z}$ for $\mu = const$ (real if the dipole is parallel to the x-axis)

Complex potential of a vortex flow around the origin: $f(z) = \frac{i\Gamma}{2\pi} \ln(z)$ for $\Gamma = const$ (always real)

Force (per unit length) on a cylinder in a flow with far-distance velocity U: $F_x - iF_y = i\rho U \Gamma$

Definition of the circulation around a closed curve γ is given by $\Gamma = \oint_\gamma \mathbf{v} \cdot \mathbf{ds}$ where the line integral is taken in counterclockwise direction

Useful values

gravitational acceleration: $g \approx 10m/s^2$

$\sin(6°) \approx 0.1$

$\cos(6°) \approx 1$

$\sin(12°) \approx 0.2$

$\cos(12°) \approx 0.98$